

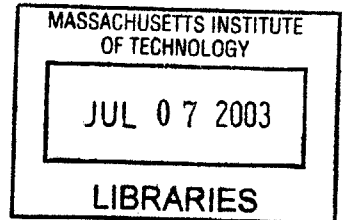
# Efficient Signal, Code, and Receiver Designs for MIMO Communication Systems

by

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**BARKER**



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## Abstract

The so-called diversity-multiplexing tradeoff characterizes the fundamental interaction between the robustness and capacity gains obtainable from multiple-input and multiple-output (MIMO) systems in fading environments. This thesis develops practical schemes for approaching the optimal tradeoff in various delay and complexity regimes. We focus on a two-transmit and two-receive antenna system, in which the receiver has channel knowledge, but the transmitter does not.

We first investigate uncoded transmission. We propose a class of *lattice-reduction-aided* low-complexity detectors that can achieve near maximum likelihood performance and the best diversity-multiplexing tradeoff achievable by any length-one code.

We also design a family of structured space-time block codes that we call *tilted-QAM codes*. It achieves the optimal infinite-delay tradeoff with the necessary minimum delay of two, answering a previously open question. It uses constellation rotation ideas to effectively spread information across space and time. We identify rotation angles that are universally optimal at all rates in terms of a determinant criterion.

We further develop efficient coding schemes using long error correction codes. In particular, we combine them with tilted-QAM codes using hard and soft decision decoding to obtain good performance at moderate SNR. These new systems are compared to orthogonal space-time coded systems, which we show to achieve near optimal performance at low SNR. We also examine traditional sequential versions and develop new block versions of the Bell Labs layered architecture (BLAST). While some of these can in principle reach the performance limit at all SNRs, we show they also have various practical problems.

Finally, for the case where no channel knowledge is available, we present a geometric view of the signal design problem. This view reveals how training based approaches can achieve the optimal (non-coherent) diversity-multiplexing tradeoff.

Thesis Supervisor: Gregory W. Wornell  
Title: Professor





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# Contents

<b>1</b>	<b>Introduction</b>	<b>19</b>
1.1	Channel and System Model . . . . .	20
1.2	Thesis Outline . . . . .	23
<b>2</b>	<b>Theoretical Background</b>	<b>27</b>
2.1	Channel Capacity and Outage Probability . . . . .	27
2.2	Visualizing Rate and Robustness Gains . . . . .	28
2.3	Diversity-Multiplexing Tradeoff . . . . .	31
2.3.1	Definitions . . . . .	31
2.3.2	Optimal Tradeoff Results . . . . .	32
2.3.3	Two-Transmit Two-Receive Antenna Case . . . . .	34
2.3.4	Visualizing The Tradeoff . . . . .	37
2.3.5	Local Diversity-Multiplexing Tradeoff . . . . .	40
2.4	Error Probability and Design Criteria . . . . .	41
2.5	Performance of Gaussian Random Codes . . . . .	45
2.5.1	Tradeoff Achieved . . . . .	46
2.5.2	Worst-Pair Bound . . . . .	47
<b>3</b>	<b>Uncoded Systems and Efficient Detection</b>	<b>51</b>
3.1	Introduction . . . . .	51
3.2	Traditional Detectors . . . . .	52
3.3	Lattice Reduction . . . . .	56
3.3.1	Choice of optimal basis . . . . .	57

3.3.2	Reduction Algorithm . . . . .	59
3.3.3	Convergence and Complexity . . . . .	61
3.4	Gaussian Channels . . . . .	64
3.4.1	Complexity . . . . .	64
3.4.2	Performance . . . . .	64
3.5	Rayleigh Fading Channels . . . . .	67
3.5.1	Complexity . . . . .	67
3.5.2	Performance . . . . .	67
3.5.3	Diversity-Multiplexing Tradeoff . . . . .	70
3.6	Lattice Reduction at Transmitter . . . . .	72
3.7	Higher Dimensional Lattice Reduction . . . . .	76
3.7.1	Existing Algorithms . . . . .	76
3.7.2	Complexity and Performance of LLL . . . . .	79
3.8	Summary . . . . .	82
<b>4</b>	<b>Structured Codes with Minimum Delay</b>	<b>85</b>
4.1	Introduction . . . . .	85
4.2	OSTBC . . . . .	86
4.2.1	The Smart Repetition . . . . .	87
4.2.2	Theoretical Performance Analysis . . . . .	87
4.2.3	Simulation Results . . . . .	89
4.3	Tilted-QAM Code . . . . .	90
4.3.1	The Rotation Design . . . . .	90
4.3.2	Choice of rotation angles . . . . .	92
4.4	Theoretical Performance Analysis . . . . .	95
4.4.1	Minimum Distance Property . . . . .	97
4.4.2	Determinant Counting . . . . .	101
4.4.3	Determinant Counting: Higher Dimensional Cases . . . . .	103
4.5	Simulation Results . . . . .	106
4.5.1	ML/Sphere Decoding . . . . .	107

4.5.2	Lattice Decoding . . . . .	109
4.6	Tilted-QAM in Single Antenna Case . . . . .	110
4.6.1	Channel Model and Theoretical Background . . . . .	111
4.6.2	Tilted-QAM design . . . . .	113
4.6.3	Error Probability Evaluation . . . . .	114
4.7	Summary . . . . .	117
<b>5</b>	<b>Error Correction Code Enhanced Systems</b>	<b>119</b>
5.1	Introduction . . . . .	119
5.2	OSTBC . . . . .	121
5.2.1	Equivalent channel . . . . .	122
5.2.2	Achievable Performance . . . . .	123
5.3	Diagonal-BLAST . . . . .	126
5.3.1	Layered Encoding . . . . .	126
5.3.2	Layered Decoding . . . . .	128
5.3.3	D-BLAST Caveats . . . . .	132
5.3.4	Experimental Setup . . . . .	138
5.3.5	Simulation Results . . . . .	141
5.4	Modified BLAST in Block Form . . . . .	143
5.4.1	Code Designs . . . . .	144
5.4.2	Multiple Access Channel Framework . . . . .	146
5.4.3	V-BLAST . . . . .	151
5.4.4	Two-Layer-D-BLAST . . . . .	154
5.4.5	X-BLAST . . . . .	157
5.4.6	Comparisons . . . . .	159
5.5	Tilted-QAM With Hard-Decision ECC . . . . .	163
5.5.1	System Setup . . . . .	164
5.5.2	Simulation Results . . . . .	165
5.6	Tilted-QAM with Soft-Decision ECC . . . . .	167
5.6.1	System Setup . . . . .	168

5.6.2	Iterative Soft-Decision Decoder . . . . .	169
5.6.3	Simulation Results . . . . .	173
5.7	Summary . . . . .	175
<b>6</b>	<b>Non-Coherent Communications</b>	<b>177</b>
6.1	Theoretical Background . . . . .	178
6.1.1	Capacity . . . . .	178
6.1.2	Capacity Achieving Distribution . . . . .	180
6.2	Non-Coherent Communication Signal Design . . . . .	181
6.2.1	Design Criterion . . . . .	182
6.2.2	Existing Schemes . . . . .	183
6.3	Geometric Approach . . . . .	185
6.3.1	Projection Matrices . . . . .	185
6.3.2	Embedding on Spheres . . . . .	186
6.3.3	Signal Design . . . . .	187
6.3.4	Relationship to Training . . . . .	189
6.4	Channel Training Approach . . . . .	190
6.4.1	Quality of Channel Estimation . . . . .	191
6.4.2	Effect of Imperfect Channel Knowledge . . . . .	192
<b>7</b>	<b>Summary and Future Directions</b>	<b>195</b>
7.1	Contributions . . . . .	195
7.2	Future Directions . . . . .	198
7.2.1	Coherent Communications . . . . .	198
7.2.2	Non-coherent Communications . . . . .	199

# List of Figures

1-1	Multiple antenna channel with $N_t$ transmit and $N_r$ receive antennas.	20
2-1	Using multiple antennas allows increased data rate. . . . .	30
2-2	Using multiple antennas allows increased robustness or diversity. . . .	30
2-3	Optimal diversity-multiplexing tradeoff curve $d_{\text{out}}(r)$ for a system with $N_t$ transmit antennas and $N_r$ receive antennas. . . . .	33
2-4	Optimal diversity-multiplexing tradeoff curve $d_{\text{out}}(r)$ for the two-transmit two-receive antenna case. . . . .	35
2-5	Family of outage probability curves as functions of SNR for various target rates $R$ in the $N_t = N_r = 2$ case. . . . .	38
2-6	As rate grows with SNR, i.e., $R = r \log_2(\text{SNR})$ , outage probability $P_{\text{out}}(R, \text{SNR})$ decays with SNR with slope $d(r)$ . . . . .	38
2-7	Linearized approximation of Figure 2-5, which clearly shows two regions of the $P_{\text{out}}$ -SNR space with different slopes of curves and horizontal spacings between curves. . . . .	39
2-8	Linearized approximation of Figure 2-6. . . . .	39
2-9	Diversity-multiplexing tradeoff achieved using Gaussian random codes of various lengths. Optimal tradeoff is achieved with $T \geq 3$ . $T = 2$ codes (with expurgation) can achieve the end points, but is sub-optimal for $0 < r < 1$ . $T = 1$ codes only achieve a maximum diversity of $d = 2$ when $r = 0$ , which is the most any length one code can do. . . . .	47

2-10	The upper bound on the diversity-multiplexing tradeoff achievable using Gaussian random codes based on the worst-pair error probability, $d(r) < 2d_{\text{out}}^{-1}(rT)$ . Short codes with $T \leq 2$ are sub-optimal due to the worst-pair being particularly bad. . . . .	49
3-1	Comparison of decision boundaries for various detection methods. . .	55
3-2	Using lattice reduction in conjunction with traditional detectors. . . .	57
3-3	Number of iterations needed to find the optimal reduced basis. $\mathbf{b}_1$ is fixed at $[1 \ 0]^T$ , each entry of $\mathbf{b}_2$ ranges from 0 to 1 in 0.01 increments.	63
3-4	Comparison of the decision regions for MLD and LR-ICD. Minimum distances to the decision boundaries are also compared. . . . .	65
3-5	Distribution of number of iterations needed for $2 \times 2$ lattice reduction.	68
3-6	Symbol error rate curves for various detection methods in the $2 \times 2$ complex case. The constellation used is 16-QAM. . . . .	69
3-7	Comparisons of the cumulative density of $d_{\text{min}}^2$ . . . . .	70
3-8	As constellation size grows, the gap between the symbol error rates of MLD and LR-BLAST diminishes. The noise level is such that the SNR is 25 dB for the 16-QAM constellation. . . . .	71
3-9	Uncoded system with LR-BLAST decoder. The maximum slope reached is 2. The horizontal spacings between the curves are 6 dB. . . . .	72
3-10	When the transmitter knows the channel, it can pre-compensate for the distortion by transmitting $\mathbf{H}^{-1}\mathbf{x}$ , so that the received constellation is the original one. . . . .	73
3-11	At the transmitter, all points that are congruent modulo the constellation region, which form a lattice, are use to represent the same message. Points labeled “o” represent the same message as the point labeled “+”. At the receiver, any “o” can be mapped back to “+” via modulo operations. . . . .	74



3-12	Treat the original transmitted constellation region, $\mathbf{H}^{-1}\mathbf{x}$ , as a unit cell of a lattice. Power reduction can be achieved by using a more square unit cell corresponding to a different basis as the transmitted constellation region. Regions shaded in the same way and labeled using the same number are congruent to each other and represent the same set of messages. . . . .	75
3-13	Empirical distribution of number of iterations needed for $n \times n$ (real) lattice reduction using the LLL algorithm for the cases of $n = 2, 4, 6, 8$ dimensions. . . . .	81
3-14	Empirical cumulative distribution, indicating probability of needing $x$ iterations or more. . . . .	81
3-15	Performance of LR-BLAST detectors using the LLL algorithm, compared to that of the ML detector, for $n = 2, 4, 6, 8$ dimensional cases. The ratio $d_{\min}^{\text{LR-BLAST}}/d_{\min}^{\text{ML}}$ indicates how far LR-BLAST is away from optimal. . . . .	83
3-16	Performance of LR-ICD detectors using the LLL algorithm, compared to that of the ML detector, for $n = 2, 4, 6, 8$ dimensional cases. . . . .	83
4-1	Diversity-multiplexing tradeoff achieved by orthogonal space-time block code, compared with the optimal tradeoff and that of the expurgated Gaussian random code, for the case $N_t = N_r = T = 2$ . . . . .	88
4-2	Error rate curves of OSTBC (dark) and outage probability curves (light) for various rates. We see that the maximum diversity of four is achieved, but there is a loss of multiplexing gain. . . . .	90
4-3	Rotate $(s_{11}, s_{22})$ to obtain $(x_{11}, x_{22})$ , so that each non-zero information symbol pair $(s_{11}, s_{22})$ leads to both non-zero $x_{11}$ and non-zero $x_{22}$ and effectively appear in both rows and columns of the codeword matrix $\mathbf{X}$ . . . . .	91
4-4	Maximize the minimum $ 2 \det(\mathbf{X}) $ as a function of $2\theta_1$ and $2\theta_2$ for the case where $s_{ij}$ each takes the value of 0 and 1. . . . .	93

4-5	Worst-case determinant as a function of $\theta_1$ , while $\theta_2 = \pi/4 - \theta_1$ . As constellation size increases, although the optimal value of $\theta_1$ remains at $\arctan(1/2)/2$ , the sensitivity increases. Slight deviation of $\theta_1$ from its optimal value significantly reduces the resulting worst-case determinant.	96
4-6	Growth rate of the number of matrices with a particular determinant as a function of the constellation size $M$ .	104
4-7	Error rate curves of the proposed titled-QAM code (dark) and the outage probability curves (light) for various rates. We see that the two sets of curves have similar slopes and horizontal gaps, which means that they have similar diversity and multiplexing gains.	107
4-8	Tilted-QAM encoding with lattice-reduction-aided BLAST decoding. The maximum slope reached is only 2. The gaps between the curves are 6 dB, indicating full multiplexing gain.	110
4-9	Single antenna fading channel over two channel realizations.	112
4-10	The sum $\sum_{\substack{-M < a, b < M \\ (a,b) \neq (0,0)}} f(a, b)$ as a function of $M$ , the number of times $f(a, b) = 1$ , and their approximations (dash).	117
5-1	OSTBC effectively transforms a $2 \times 2$ multiple antenna channel to two independent AWGN channels with identical gains $\ \mathbf{H}\ $ .	123
5-2	Concatenation of an OSTBC inner code with an error correction outer code.	124
5-3	Comparison of the family of channel outage probability curves (solid) and the family of OSTBC outage probability curves (dash) as functions of SNR for rates $2^{-5}, 2^{-4}, \dots, 1, 2, 4, 6, \dots, 18, 20$ .	125
5-4	BLAST encodes in diagonal layers labeled with different alphabetical letters.	127
5-5	BLAST-nulling decoding scheme. Interference from symbols in later layers, which are not yet decoded, are nulled out via QR factorization of the channel matrix. Interference from symbols in previous layers, which are already decoded, are eliminated using successive cancellation.	130

5-6	BLAST-MMSE effectively transforms a $2 \times 2$ multiple antenna channel to two independent AWGN channels with effective gains $r_{11}$ and $\sqrt{r_{22}^2 + r_{12}^2/(1 + \rho r_{11}^2)}$ . . . . .	132
5-7	Demonstration of the finite constellation size problem. When the constellation size used is too small, there is a loss of diversity gain. . . .	137
5-8	If we select the constellation size to be $M = \log_2(1 + \text{SNR})$ , then the outage probability associated with BLAST (dash) seems to be very close to the ultimate channel outage probability (solid). The loss due to finite constellation effect is small. . . . .	138
5-9	Gray-labeling with 8-PAM constellation. . . . .	139
5-10	Approximations of log likelihood ratios of different bits as functions of $y = x + w$ for an 8-PAM constellation with $\sigma_w^2 = 1$ . . . . .	141
5-11	Block error rate for $R = 6$ and $R = 8$ b/s/Hz using D-BLAST-MMSE architecture on a two-transmit two-receive antenna system. . . . .	142
5-12	V-BLAST, where coding is restricted to one row of the transmitted signal matrix. . . . .	145
5-13	Two layers of BLAST. Both layers are end layers. . . . .	145
5-14	X-BLAST, where two codewords cross like in OSTBC or tilted-QAM. . . . .	146
5-15	Capacity region for a multiple access channel. . . . .	147
5-16	The achievable rate region has two sub-regions. The darkly shaded one requires joint decoding. . . . .	148
5-17	The four different ways in which the $R_1 = R_2$ line can intersect the achievable rate bounds. . . . .	149
5-18	Diversity-multiplexing tradeoffs achieved by V-BLAST encoding with joint and separate decoding. . . . .	152
5-19	Outage probability curves for rates $2, 4, \dots, 20$ b/s/Hz achieved by V-BLAST encoding with joint decoding (thick dashed), comparing with that of channel outage probability (thin solid). . . . .	153

5-20	Outage probability curves for rates $2, 4, \dots, 20$ b/s/Hz achieved by V-BLAST encoding with more practical separate decoding based on successive cancellation (thick dashed), comparing with that of channel outage probability (thin solid). . . . .	153
5-21	Diversity-multiplexing tradeoffs achieved by two-layer-D-BLAST encoding with joint and separate decoding. . . . .	156
5-22	Outage probability curves for rates $2, 4, \dots, 20$ b/s/Hz achieved by two-layer-D-BLAST encoding with joint decoding (thick dashed), comparing with that of channel outage probability (thin solid). . . . .	158
5-23	Outage probability curves for rates $2, 4, \dots, 20$ b/s/Hz achieved by two-layer-D-BLAST encoding with more practical separate decoding (thick dashed), comparing with that of channel outage probability (thin solid). . . . .	158
5-24	Diversity-multiplexing tradeoffs achieved by X-BLAST encoding with joint and separate decoding. . . . .	159
5-25	Outage probability curves for rates $2, 4, \dots, 20$ b/s/Hz achieved by X-BLAST encoding with more practical separate decoding (thick dashed), comparing with doing joint decoding, which is also the channel outage probability (thin solid). . . . .	160
5-26	Gaps to capacity as a function of rate at $P_{\text{out}} = 10^{-3}$ for various systems. . . . .	161
5-27	Diversity-multiplexing tradeoff curves achieved by variations of BLAST in block form and OSTBC. . . . .	162
5-28	Concatenation of a tilted-QAM inner code with a Reed-Solomon outer code. . . . .	164
5-29	Block error rate curves for 16-QAM, 64-QAM, and 256-QAM cases. As we gradually reduce the data rate by 1 b/s/Hz, the block error rate lowers due to stronger coding. However, the gain diminishes. . . . .	166

5-30	Block error rate curves for 16-QAM, 64-QAM, and 256-QAM cases with 1 b/s/Hz rate reduction using RS coding. The unmarked curves are the corresponding channel outage probability curves. . . . .	167
5-31	Concatenation of a tilted-QAM inner code with an LDPC outer code with a two component iterative soft-decision decoder. . . . .	168
5-32	Passing of bit-wise LLR scores between an LDPC decoder and a lattice-aware detector unit consisting of a lattice detector and an MMSE detector. . . . .	170
5-33	Block error rates achieved by tilted-QAM-LDPC concatenated systems (thick solid), compared with D-BLAST-MMSE (dashed), and the ultimate outage probability limit (thin solid), at $R = 6$ and $R = 8$ b/s/Hz, using two-transmit two-receive antenna systems. . . . .	173
6-1	In the space of symmetric matrices, all projection matrices (of any rank) are embedded on (the surface of) a sphere centered at $\mathbf{I}_T/2$ with radius $\sqrt{T}/2$ . Projection matrices with a particular trace (rank) are embedded on lower dimensional spheres. This figure is from [5]. . . .	188
6-2	Using a “polygon” approximation to design a set of well separated points on a sphere. . . . .	189



# Chapter 1

## Introduction

Over the past few years, it has been shown that using multiple antennas can significantly increase the capacity and robustness of communication systems in fading environments. Capacity grows with the number of antennas used. Approximately twice the amount of information can be communicated using two transmit antennas and two receive antennas, without spending any extra time, bandwidth, nor power. In a fading environment, the channel quality may vary due to, for example, movement of the transmitter or the receiver. In such an environment, using multiple antennas makes it less likely for the channel to be in a deep enough fade such that the transmitted information can not go through. This is because the multiple links between the multiple antennas provide us with multiple opportunities and more protection.

Since the benefits of using multiple antennas have been recognized, much work have been done toward designing coding and decoding schemes to realize these gains promised by theoretical studies. However, some of the studies focus only on the robustness gain but do not capitalize on the capacity gain; while others concentrate on the capacity gain but have less than optimal robustness.

More recently, there are efforts on realizing both capacity and robustness gains simultaneously. Zheng and Tse [41] established that there is a tradeoff between these two types of gains, i.e., how fast error probability can decay and how rapidly data rate can increase with signal to noise ratio (SNR). Furthermore, they analytically evaluated the efficient frontier of this *diversity-multiplexing tradeoff* for systems with any

number of transmit and receive antennas and showed that the frontier is achievable using sufficiently long Gaussian random codes.

In this thesis, our goal is to design practical multiple antenna systems aiming at achieving the optimal diversity-multiplexing tradeoff. We design structured linear signaling and coding schemes at the transmitter with easy implementation, and design corresponding decoding algorithms at the receiver with moderate computational complexity. We demonstrate the performance of our designs through both theoretical analysis and numerical simulations.

## 1.1 Channel and System Model

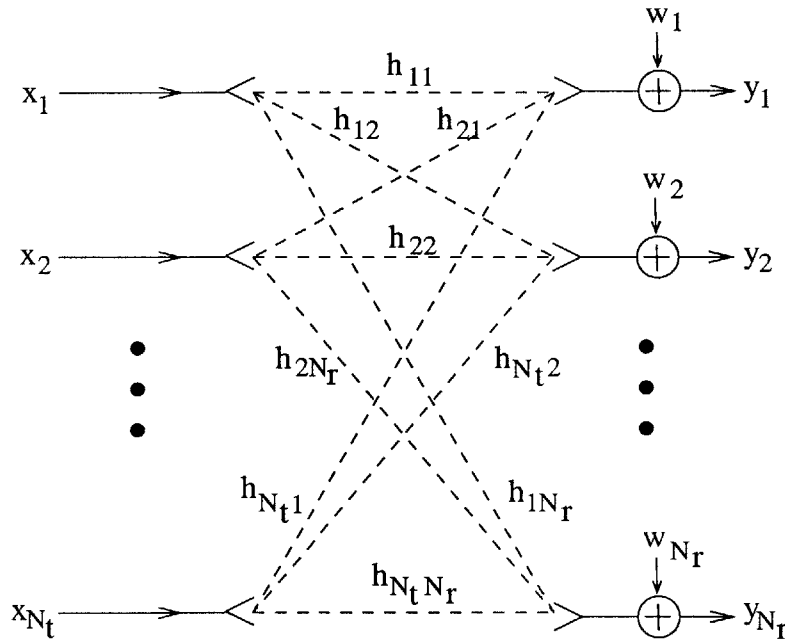


Figure 1-1: Multiple antenna channel with  $N_t$  transmit and  $N_r$  receive antennas.

Let us first describe the channel and system model. Figure 1-1 shows a communications link with  $N_t$  transmit antennas and  $N_r$  receive antennas. At each time instant,  $N_t$  signals,  $(x_1, x_2, \dots, x_{N_t})$ , satisfying an average power constraint, are transmitted using  $N_t$  antennas. Each of them reaches all  $N_r$  receive antennas.

In this thesis, we model the channel as flat, Rayleigh, and block fading, with channel knowledge at the receiver, as well as additive white Gaussian noise (AWGN).



These models are commonly used in the multiple antenna communications literature and have been proven useful in practice. We also restrict our attention to the case where there are at least as many receive antennas as transmit antennas, i.e.,  $N_r \geq N_t$ .

- **Flat Fading:** We model each wireless link between each pair of transmit and receive antennas as a simple scaling by channel gain  $h_{ij}$ . This is valid when the signal bandwidth is narrow enough so that the entire spectrum experiences the same fading coefficient.
- **Rayleigh Fading:** We model the statistics of the random channel coefficients,  $h_{ij}$ , using the *Rayleigh fading* model. This means that they are independent and identically distributed (IID) with zero mean, unit variance, circularly symmetric, complex Gaussian density,  $\mathcal{CN}(0, 1)$ . It is worth noting that this model is often used because it leads to more tractable theoretical analysis, but it is not entirely accurate. In most environments, the channel coefficients are correlated. The correlation is less when the antennas are well separated and there are a large number of scatters in the environment. Given a particular environment, only a certain number of antennas can be used before the channel coefficients become too correlated and the channel model breaks down. For example, for indoor environments, only up to three to eight antennas can be used [27]. Therefore, practically speaking, we can not indefinitely increase the number of antennas used and hope to obtain arbitrarily large capacity and robustness gains.
- **Block Fading:** We model the time varying nature of the channel using block fading, meaning that the channel stays fixed for a certain period, call the *coherence time* of the channel, and then changes to something independent for the next block. In reality, channel coefficients changes gradually from one time instant to the next. However, this is hard to analyze. Therefore, block fading model is often used for its simplicity.
- **Channel Knowledge at Receiver:** For most of this thesis, we assume that perfect channel knowledge is available at the receiver but not at the transmitter,

i.e., *coherent detection*. Practically speaking, the receiver can not know the channel perfectly. However, if the channel varies slowly, we can assume that the receiver has sufficient time to get a good estimate of the channel. Again, this assumption is not entirely accurate but makes our problem easier. In Chapter 6, we explore the scenario where no channel knowledge is available, which we call *non-coherent detection*. This happens when the channel varies too fast and is difficult to track. We always assume that transmitter does not have knowledge of the channel, because this requires feedback from the receiver.

- **AWGN at Receiver:** At each receiver, signals received from all transmit antennas are added together, along with an IID additive white (complex) Gaussian noise with zero mean and variance per dimension  $\sigma_w^2$ , i.e.,  $\mathcal{CN}(0, 2\sigma_w^2)$ .

We also restrict our attention to the case where the code duration, denoted by  $T$ , is shorter than the coherence time of the channel, so that each codeword experiences only one channel realization. The system we design can serve as a building block to build more complex systems where coding happens over multiple channel realizations through interleaving in either time or frequency or both.

With the above channel models, we can express the multiple antenna channel (over one channel realization) mathematically as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}, \quad (1.1)$$

where  $\mathbf{H}$  is the  $N_r \times N_t$ ,  $N_r \geq N_t$ , multiple antenna channel,  $\mathbf{X}$  is the  $N_t \times T$  transmitted signal matrix,  $\mathbf{W}$  represents the additive white Gaussian noise, and  $\mathbf{Y}$  is the received signal matrix. Written in a matrix form, we have

$$\begin{bmatrix} y_{11} & \cdots & y_{1T} \\ y_{21} & \cdots & y_{2T} \\ \vdots & \ddots & \vdots \\ y_{N_r 1} & \cdots & y_{N_r T} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1N_t} \\ h_{21} & \cdots & h_{2N_t} \\ \vdots & \ddots & \vdots \\ h_{N_r 1} & \cdots & h_{N_r N_t} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1T} \\ x_{21} & \cdots & x_{2T} \\ \vdots & \ddots & \vdots \\ x_{N_t 1} & \cdots & x_{N_t T} \end{bmatrix} + \begin{bmatrix} w_{11} & \cdots & w_{1T} \\ w_{21} & \cdots & w_{2T} \\ \vdots & \ddots & \vdots \\ w_{N_r 1} & \cdots & w_{N_r T} \end{bmatrix}. \quad (1.2)$$

Let the energy constraint on the transmitted signal be such that each dimension of  $x_{ij}$  has an average energy of  $E_s$ . The total transmit SNR over all antennas is thus

$$\text{SNR} = N_t \frac{E_s}{\sigma_w^2}. \quad (1.3)$$

The per-antenna SNR is

$$\rho = \frac{\text{SNR}}{N_t} = \frac{E_s}{\sigma_w^2}. \quad (1.4)$$

Note that each column of the transmitted signal matrix  $\mathbf{X}$  corresponds to what is transmitted at one time by multiple antennas; and each row corresponds to what one antenna transmits over time. When we perform coding across rows of  $\mathbf{X}$ , we refer to it as coding across space. Coding across columns is referred to as coding across time. When the transmission rate is  $R$  b/s/Hz, there are  $2^{RT}$  codeword matrices  $\mathbf{X}$  to be designed.

## 1.2 Thesis Outline

We first review some theoretical background on multiple antenna communications in Chapter 2. We present the channel capacity formula and define the ultimate performance limit, the outage probability. We then review the diversity-multiplexing tradeoff definition and the optimal tradeoff result obtained by Zheng and Tse [41], and provide some of our own interpretations. Next, we look at what determines the error probabilities of a given coding scheme, and from which we obtain some code design rules. We then analyze Gaussian random codes as a benchmark and see that sufficiently long Gaussian random codes can achieve the optimal diversity-multiplexing tradeoff while short ones can not due to particularly bad randomly selected codeword pairs.

In this thesis, our goal is to design practical multiple antenna systems aiming at achieving the optimal diversity-multiplexing tradeoff. we focus our research on the two-transmit two-receive antenna system, which arises frequently in practice, and can

lead to important insights on how to build larger systems with more antennas. We study the design problem in various delay and complexity regimes.

In Chapter 3, we investigated the case of uncoded transmission with zero delay, i.e., code duration  $T = 1$ . We propose low-complexity detectors that can achieve near maximum likelihood performance by operating traditional detectors in a reduced lattice basis. We identify the optimal basis to operate in and describe an iterative algorithm for finding it. Using these improved detectors, the uncoded system achieves the best diversity-multiplexing tradeoff achievable by any length-one code.

In Chapter 4, we move on to the case of coding with the minimum delay necessary for achieving the optimal diversity-multiplexing tradeoff. We construct a family of short structured space-time block codes for the two-transmit two-receive antenna system. It achieves the optimal diversity-multiplexing tradeoff and has the minimum delay of two necessary for optimality. It is a modification of the well-known orthogonal space-time block codes (OSTBC) [1, 34], which uses a smart repetition to achieve the maximum diversity gain at the expense of multiplexing gain. We use an idea of rotation, instead of repetition, of cross-diagonal entries of an uncoded transmission to achieve spreading of information across space and time to obtain maximum diversity while preserving multiplexing gain. Rotation angles that are optimal in terms of a determinant criterion and universal for all rates are identified. We refer to this code construction as the *tilted-QAM code*.

In Chapter 5, we experiment with further enhancing system performance using powerful error correction codes (ECC). The goal is to understand how to build practical systems with good performance. We study several coding systems. We show that an system based on OSTBC can achieve near optimal performance in the low SNR regime. We then describe the Bell labs layered space-time (BLAST) architecture and show that it has the potential to achieve channel capacity but has practical problems. We also present and analyze several variations of the BLAST. Finally, we explore the possibility of combining hard and soft decision error correction coding with the tilted-QAM code.

In Chapter 6, we explore the case where channel knowledge is available at neither

the transmitter nor the receiver. We first review some existing theoretical results on non-coherent multiple antenna communications, and then discuss the problem of signal design. We present evidence that the channel training approach could lead to good diversity-multiplexing tradeoff.

In Chapter 7, we summarize the contributions of this thesis and discuss future research directions.



# Chapter 2

## Theoretical Background

In this chapter, we first review the channel capacity formulation and the concept of outage probability, which sets the ultimate performance limit. In section 2.2, we illustrate the capacity and robustness gains that can be potentially obtained using multiple antennas. In section 2.3, we review the diversity-multiplexing tradeoff framework and provide additional intuition. In section 2.4, we derive error probability expressions for evaluating coding schemes and obtain criteria for good codes. In section 2.5, we use the formulation from section 2.4 to examine the performance of Gaussian random codes of different lengths. We explain why short Gaussian random codes can not achieve the optimal diversity-multiplexing tradeoff.

### 2.1 Channel Capacity and Outage Probability

Given a particular channel realization  $\mathbf{H}$ , the theoretical limit of the amount of data we can transmit through the channel reliably, i.e., with arbitrarily low error rate, is the channel capacity [36],

$$C_{\text{channel}}(\mathbf{H}, \rho) = \log_2(\det(\mathbf{I}_{N_r} + \rho\mathbf{H}\mathbf{H}^\dagger)) \text{ b/s/Hz}, \quad (2.1)$$

where  $\rho = \text{SNR}/N_t$  is the average transmit SNR per antenna and  $\det(\cdot)$  denotes the determinant function. This data rate is achievable using infinitely long codes with

unlimited complexity. We note that the input distribution used is  $\mathcal{CN}(0, \rho)$ . Since, the transmitter has no knowledge of the channel, this distribution is a reasonable default choice. If the channel were known, it would be possible to choose a better input distribution.

Another important concept is *outage*. In our system model, coding is performed over one channel realization, and since the channel is a random matrix, the realized channel capacity is a random variable. Since the transmitter has no knowledge of the channel, it can not adjust the data rate according to the realized channel and must transmit at a fixed rate  $R$  b/s/Hz. Therefore, when the realized channel capacity is below  $R$ , the receiver can not decode even with powerful codes. This is the outage event, and the outage probability is

$$P_{\text{out}}(R, \rho) = P[C(\mathbf{H}, \rho) < R]. \quad (2.2)$$

This is the ultimate performance limit when coding is done over only one channel realization.

Achieving the outage probability requires using infinitely long and complex codes. In practice, long codes leads to large delay and high complexity requires expensive hardware. Therefore, they are usually not satisfied in practice, and we must content with finite delay and moderate complexity coding schemes. The capacity formulas can be used as performance limits and help us evaluate practical systems.

## 2.2 Visualizing Rate and Robustness Gains

Next, let us use the capacity and outage probability formulation to gain some insight into how capacity and robustness gains can be obtained using multiple antennas.

In the single antenna case, which is simply the AWGN channel ( $y = hx + w$ ), the well-known channel capacity, originally derived by Shannon, is

$$C_{\text{channel}}(\mathbf{H}, \rho) = \log_2 (\rho |h|^2 + 1). \quad (2.3)$$



For two-transmit two-receive antenna systems, the channel capacity is

$$C_{\text{channel}}(\mathbf{H}, \rho) = \log_2 (\rho^2 |\det(\mathbf{H})| + \rho(|h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2) + 1) \quad (2.4)$$

At high SNR, assuming  $\det(\mathbf{H}) \neq 0$ , the  $\rho^2$  term dominates, and the channel capacity grows like  $2 \cdot \log_2(\rho)$ , compared to the  $1 \cdot \log_2(\rho)$  in the single antenna case. This shows the capacity gain due to having multiple antennas.

While the  $\rho^2$  term can lead to large channel capacity, the linear term prevents the capacity from becoming too small. Because it is the sum of the energy of all entries of  $\mathbf{H}$ , all four terms has to be small for the total to be small. This makes the channel more robust toward fading of individual channel coefficients.

Let us visualize the potential rate and robustness gains due to using multiple antennas by comparing the achievable rates and outage probabilities for systems with one, two, four, and eight antennas at the transmitter and equal number of antennas at the receiver.

Figure 2-1 shows a plot of achievable data rate vs. SNR when the target outage probability is fixed at 1%. Starting from the lowest curve for the single antenna case, every time the number of antennas is doubled, the achievable data rate is also approximately doubled. The slopes of the curves approach  $N$  bits per 3 dB increase in SNR, where  $N$  is the number of antennas. This demonstrates the capacity gain.

Figure 2-2 shows a plot of the outage probability vs. SNR when the target data rate is set at 1 bit per dimension, or 2 bits per antenna. Starting from the top curve for the single antenna case, every time the number of antennas is doubled, the slope of the curve increases. The limiting slope is 1 for the top curve and 4 for the second one. In fact, the limiting slope approaches  $N^2$ . However, this is difficult to see for the lowest two curves. As a result of the increased slope, lower outage probability is achieved at the same SNR, or equivalently, lower SNR is needed to achieve the same outage probability. This demonstrates the robustness gain.

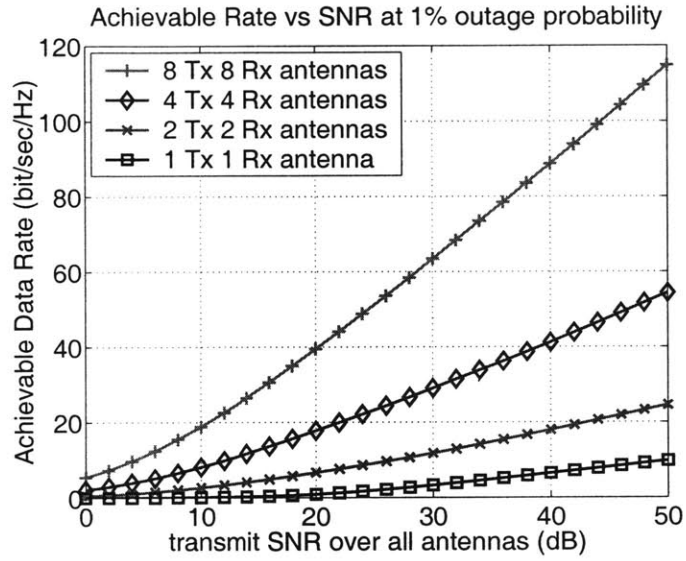


Figure 2-1: Using multiple antennas allows increased data rate.

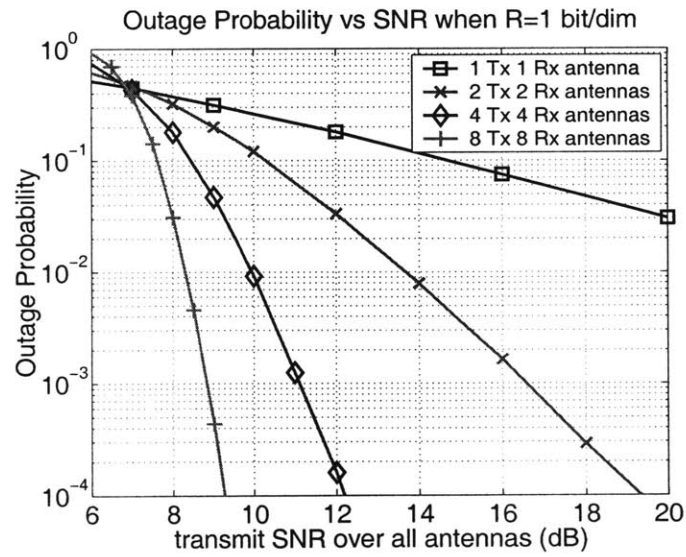


Figure 2-2: Using multiple antennas allows increased robustness or diversity.

## 2.3 Diversity-Multiplexing Tradeoff

Using multiple antennas can provide us both data rate gain as well as robustness gain toward channel fading, as we demonstrated in the last section. However, a tradeoff exists between these two types of gains; getting more of one kind requires sacrifice of the other. This tradeoff was defined and studied by Zheng and Tse in [41].

In this section, we first introduce the definition of diversity and multiplexing gains, and then review the main results on the optimal tradeoff achievable. Next, we focus on the two-transmit two-receive antenna case, examine the tradeoff analytically, as well as visualize it by plotting families of outage probability curves. Finally, we comment on local diversity-multiplexing tradeoff.

### 2.3.1 Definitions

For a given SNR, let  $R(\text{SNR})$  be the transmission rate and  $P_e(\text{SNR})$  be the error probability at that rate and SNR. Diversity gain ( $d$ ) and multiplexing gain ( $r$ ) are defined as

$$d = -\limsup_{\text{SNR} \rightarrow \infty} \frac{\log P_e(\text{SNR})}{\log \text{SNR}}, \quad (2.5)$$

and

$$r = \lim_{\text{SNR} \rightarrow \infty} \frac{R(\text{SNR})}{\log_2 \text{SNR}}. \quad (2.6)$$

Intuitively, multiplexing gain is about how fast rate increases with SNR, and diversity gain describes how fast error probability decays with SNR. If we let rate grow rapidly with SNR, error probability would not decay very fast. This is a fundamental tradeoff. This diversity-multiplexing tradeoff can be used to evaluate and compare coding schemes.

For simplicity, we use some special notations defined in [41]. We use  $\dot{=}$  to denote

exponential equality, i.e.,  $f(x) \doteq x^b$  denotes

$$\limsup_{x \rightarrow \infty} \frac{\log f(x)}{\log x} = b.$$

With this notation, diversity gain can also be written as

$$P_e(\text{SNR}) \doteq \text{SNR}^{-d}. \quad (2.7)$$

The notations  $\dot{\geq}$  and  $\dot{\leq}$  are defined similarly.

### 2.3.2 Optimal Tradeoff Results

Before looking at any particular system, let us consider the diversity-multiplexing tradeoff associated with the outage probability, i.e., replacing the error probability  $P_e(\text{SNR})$  in (2.5) with the outage probability  $P_{\text{out}}(\text{SNR})$ . When the channel is in outage, there would be a high error probability no matter what coding scheme is used. Therefore, the diversity-multiplexing tradeoff associated with the outage probability, denoted by  $d_{\text{out}}(r)$ , is an upper bound of the optimal tradeoff achievable by any system. It was shown in [41] that the tradeoff  $d_{\text{out}}(r)$  is in fact achievable using sufficiently long Gaussian random codes.

For a system with  $N_t$  transmit antennas and  $N_r$  receive antennas, Zheng and Tse evaluated  $d_{\text{out}}(r)$  in [41] and their main result is stated in the following lemma :

**Lemma 2.1** *The optimal tradeoff curve  $d_{\text{out}}(r)$  is given by the piece-wise linear function connecting the points  $(k, d_{\text{out}}(k))$ ,  $k = 0, \dots, K$  where  $K = \min(N_t, N_r)$ , and*

$$d_{\text{out}}(k) = (N_t - k)(N_r - k). \quad (2.8)$$

The function  $d_{\text{out}}(r)$  is plotted in Figure 2-3 for general values of  $N_t$  and  $N_r$ .

The tradeoff curve  $d_{\text{out}}(r)$  can be evaluated from the outage probability  $P_{\text{out}}(R, \text{SNR})$ ,

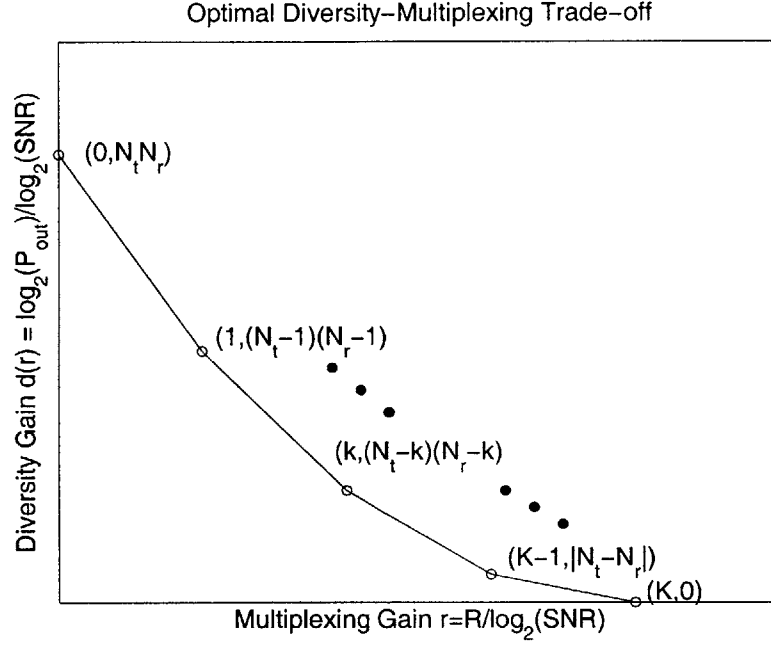


Figure 2-3: Optimal diversity-multiplexing tradeoff curve  $d_{\text{out}}(r)$  for a system with  $N_t$  transmit antennas and  $N_r$  receive antennas.

which is

$$P_{\text{out}}(R, \text{SNR}) \triangleq P \left[ \log \det (I + \text{SNR} \mathbf{H} \mathbf{H}^\dagger) < R \right]. \quad (2.9)$$

From the statistics of  $\mathbf{H}$ , whose entries are modeled using Rayleigh fading as independent and identically distributed  $\mathcal{CN}(0, 1)$  random variables, the outage probability in (2.9) can be evaluated analytically. While the exact expression is difficult to obtain, the exponential growth rate is solved in [41]. Let  $\lambda_i$  be the ordered singular values of  $\mathbf{H}$ , let  $\text{SNR}^{-\alpha_i} = |\lambda_i|^2$ , and let  $(x)^+$  denote  $\max(0, x)$ . The outage probability can be rewritten as

$$P_{\text{out}}(R, \text{SNR}) \triangleq P \left[ \prod_{i=1}^K (1 + \text{SNR} |\lambda_i|^2) < R \right] \quad (2.10)$$

$$\triangleq P \left[ \sum_{i=1}^K (1 - \alpha_i)^+ < r \right]. \quad (2.11)$$

By evaluating the probability density of  $\alpha$  and taking the limit  $\text{SNR} \rightarrow \infty$ , they obtained the following result :

**Lemma 2.2** *Let the data rate be  $R = r \log \text{SNR}$ , with  $0 \leq r \leq K = \min(N_t, N_r)$ . The outage probability*

$$P_{\text{out}}(R, \text{SNR}) = \text{SNR}^{-d_{\text{out}}(r)}, \quad (2.12)$$

where

$$d_{\text{out}}(r) = \inf_{\alpha \in \mathcal{A}'} \sum_{i=1}^K (2i - 1 + |N_t - N_r|) \cdot \alpha_i,$$

and

$$\mathcal{A}' = \left\{ \alpha \mid \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K \geq 0 \text{ and } \sum_i (1 - \alpha_i)^+ < r \right\}.$$

The resulting  $d_{\text{out}}(r)$  matches with the result of Lemma 2.1 for all  $r$ .

### 2.3.3 Two-Transmit Two-Receive Antenna Case

In most of this thesis, we focus on the two-transmit two-receive antenna case, i.e.,  $N_t = N_r = 2$ . In this case, the optimal diversity-multiplexing tradeoff curve is a piece-wise linear function connecting the points  $(0, 4)$ ,  $(1, 1)$ , and  $(2, 0)$ , as shown in Figure 2-4. Note that this curve has two linear segments.

In this section, we show a technique that allows us to quickly obtain the diversity-multiplexing tradeoff curve from the capacity expression in this  $2 \times 2$  case.

The capacity of a  $2 \times 2$  multiple antenna system is

$$\begin{aligned} C_{\text{channel}}(\mathbf{H}, \rho) &= \log_2(\det(\mathbf{I}_{N_r} + \rho \mathbf{H} \mathbf{H}^\dagger)) \\ &= \log_2(\rho^2 |\det(\mathbf{H})|^2 + \rho \|\mathbf{H}\|^2 + 1). \end{aligned}$$

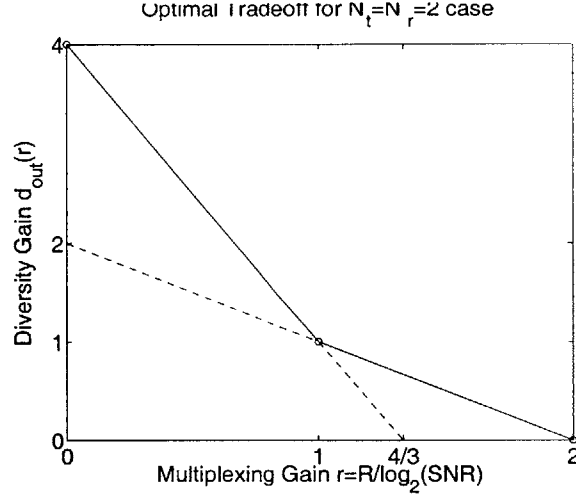


Figure 2-4: Optimal diversity-multiplexing tradeoff curve  $d_{\text{out}}(r)$  for the two-transmit two-receive antenna case.

Performing a QR factorization of  $\mathbf{H} = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$ , the above expression can be rewritten as

$$C_{\text{channel}}(\mathbf{H}, \rho) = \log_2 (\rho^2 r_{11}^2 r_{22}^2 + \rho(r_{11}^2 + |r_{12}|^2 + r_{22}^2) + 1). \quad (2.13)$$

The term  $r_{11}^2$  is the energy of the first column of  $\mathbf{H}$ , so it is the sum of the squares of four independent Gaussian random variables. Thus, it is a chi-squared random variable of order 4. Similarly,  $|r_{12}|^2$  and  $r_{22}^2$  are the energy of the second column of  $\mathbf{H}$  that are along and perpendicular to the direction of the first column. Therefore, they are chi-squared random variables of order 2. Note that for a chi-squared random variable  $\chi$  of order  $k$ ,  $P[\chi < \alpha] \doteq \alpha^{k/2}$ , for  $\alpha < 1$ .

Using  $2^R \doteq \rho^r$ , the outage probability can be written as

$$P_{\text{out}} \doteq P[\rho^2 r_{11}^2 r_{22}^2 + \rho(r_{11}^2 + |r_{12}|^2 + r_{22}^2) + 1 < \rho^r]. \quad (2.14)$$

For  $1 \leq r \leq 2$ , the first order and the constant terms are insignificant compared

to  $\rho^r$ . Therefore,

$$\begin{aligned}
P_{\text{out}} &\stackrel{\cdot}{=} P[\rho^2 r_{11}^2 r_{22}^2 < \rho^r] \\
&\stackrel{\cdot}{=} P[r_{11}^2 < 1 \text{ and } r_{22}^2 < \rho^{r-2}] \\
&\stackrel{\cdot}{=} 1 \cdot \rho^{r-2} \\
&\Rightarrow d(r) = 2 - r \text{ for } 1 \leq r \leq 2.
\end{aligned}$$

The second equality uses the fact that  $r_{22}^2$  is more likely to be small than  $r_{11}^2$  is, because  $r_{22}^2$  is a lower order chi-squared random variable. Thus, most of the time,  $r_{11}^2 r_{22}^2$  is small because  $r_{22}^2$  is small. Therefore, the event  $(r_{11}^2 < 1) \cup (r_{22}^2 < \rho^{r-2})$  is the dominant event of  $\rho^2 r_{11}^2 r_{22}^2 < \rho^r$ , resulting in the second equality.

For  $0 \leq r \leq 1$ , only the constant term in (2.14) is insignificant compared to  $\rho^r$ . Therefore,

$$\begin{aligned}
P_{\text{out}} &\stackrel{\cdot}{=} P[\rho^2 r_{11}^2 r_{22}^2 + \rho^1 (r_{11}^2 + |r_{12}|^2 + r_{22}^2) < \rho^r] \\
&\stackrel{\cdot}{=} P[r_{11}^2 < \rho^{r-1} \text{ and } |r_{12}|^2 < \rho^{r-1} \text{ and } r_{22}^2 < \rho^{-1}] \\
&\stackrel{\cdot}{=} \rho^{2(r-1)} \rho^{r-1} \rho^{-1} \\
&= \rho^{3r-4} \\
&\Rightarrow d(r) = 4 - 3r \text{ for } 0 \leq r \leq 1.
\end{aligned}$$

To obtain the second equality in this case, we use the fact that all  $r_{11}^2$ ,  $|r_{12}|^2$ , and  $r_{22}^2$  have to be less than  $\rho^{r-1}$  for the first order term to be sufficiently small. In addition, the second order term also needs to be small. To make it so, we need to have  $r_{22}^2$  to be even smaller, less than  $\rho^{-1}$ . Therefore,  $r_{11}^2 < \rho^{r-1}$ ,  $|r_{12}|^2 < \rho^{r-1}$ , and  $r_{22}^2 < \rho^{-1}$  is the dominant event of  $\rho^2 r_{11}^2 r_{22}^2 + \rho^1 (r_{11}^2 + |r_{12}|^2 + r_{22}^2) < \rho^r$ , leading to the second equality.

By looking at the outage condition (2.14) in two different regimes, the diversity-multiplexing tradeoff is obtained directly from the capacity expression. However, it becomes increasingly more difficult to apply this technique in higher dimensional cases due to the greater number of variables.



### 2.3.4 Visualizing The Tradeoff

We first visualize the relationship between SNR, rate, and outage probability by plotting  $P_{\text{out}}$  as functions of SNR for various rates  $R$  in Figure 2-5. Each curve represents how outage probability decays with SNR for a fixed rate  $R$ . As  $R$  increases, the curves move out.

Next, to see the diversity-multiplexing tradeoff for each value of  $r$ , we evaluate  $P_{\text{out}}$  as a function of SNR and  $R = r \log_2(\text{SNR})$  for a sequence of increasing SNR values, and plot a  $P_{\text{out}}(r \log_2(\text{SNR}), \text{SNR})$  curve for that  $r$ . In Figure 2-6, several such curves are plotted for various values of  $r$ ; each is labeled with the corresponding  $r$  and  $d_{\text{out}}(r)$  values. Figure 2-5 is overlaid as gray lines. For comparison purpose, dashed lines with slopes  $d_{\text{out}}(r)$  are drawn. According to Lemma 2.1, the solid and dashed curves should have matching slopes at high SNR. We see that they match quite well. From Figure 2-6, we see that when  $R$  increases faster with SNR, the corresponding outage probability decays slower. This is the fundamental diversity-multiplexing tradeoff.

To obtain further intuition, we perform the following approximation. Instead of  $P_{\text{out}}(R, \text{SNR}) \doteq \text{SNR}^{-d_{\text{out}}(r)}$ , we replace the asymptotic exponential equality  $\doteq$  with an exact  $=$ . This approximation turns the smooth  $P_{\text{out}}(R, \text{SNR})$  curves into piecewise linear lines, which would help shed more light on limiting behaviors. With the approximation, Figure 2-5 and 2-6 are re-plotted as Figure 2-7 and 2-8.

In Figure 2-8, we see that the  $P_{\text{out}}(r \log_2(\text{SNR}), \text{SNR})$  curves are now straight lines with slope  $d_{\text{out}}(r)$  exactly, which is a direct result of the approximation. In Figure 2-7, we now see a feature that is not prominent in Figure 2-5: the SNR- $P_{\text{out}}$  plane has two distinct regions, each having a set of parallel lines. The upper-right half has denser lines, while the lower-left half has more sparse and steeper lines. These two regions correspond to the two linear piece of the diversity-multiplexing tradeoff curve, as we elaborate in the next section. The boundary is the line  $P_{\text{out}} = \text{SNR}^{-1}$ , which is the line labeled  $r = 1, d = 1$  in Figure 2-8, and corresponds to the  $(1, 1)$  point (the knee) on the tradeoff.

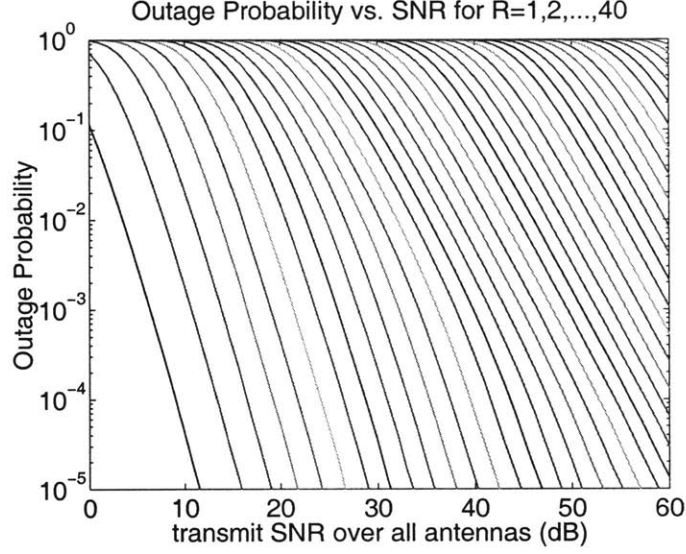


Figure 2-5: Family of outage probability curves as functions of SNR for various target rates  $R$  in the  $N_t = N_r = 2$  case.

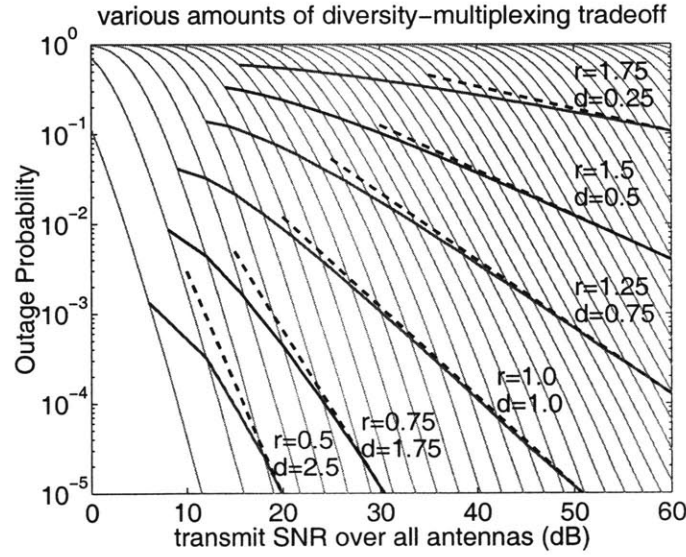


Figure 2-6: As rate grows with SNR, i.e.,  $R = r \log_2(\text{SNR})$ , outage probability  $P_{\text{out}}(R, \text{SNR})$  decays with SNR with slope  $d(r)$ .

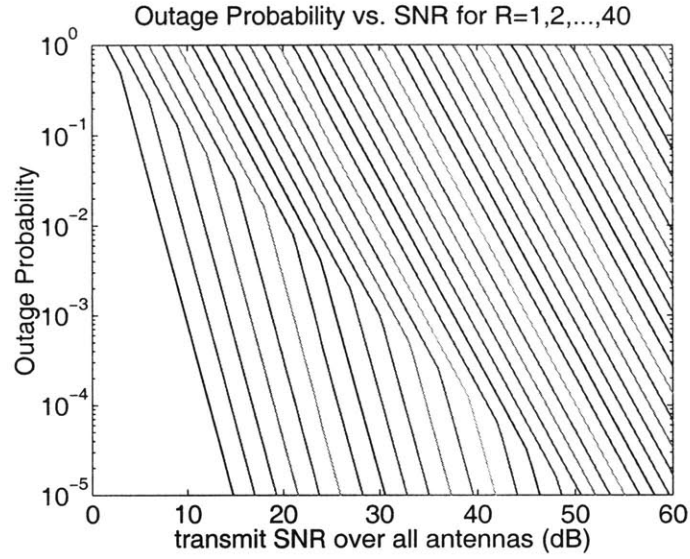


Figure 2-7: Linearized approximation of Figure 2-5, which clearly shows two regions of the  $P_{\text{out}}$ -SNR space with different slopes of curves and horizontal spacings between curves.

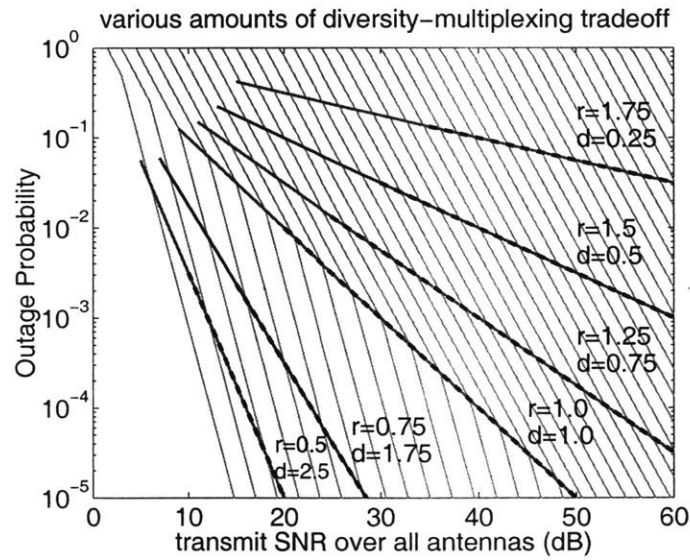


Figure 2-8: Linearized approximation of Figure 2-6.

### 2.3.5 Local Diversity-Multiplexing Tradeoff

The slopes and gaps between the curves in Figure 2-7 lead to a concept called *local* diversity-multiplexing tradeoff, which is different from the global scale tradeoff we have defined. Let us suppose that we are operating at a certain  $(R, \text{SNR}, P_{\text{out}})$  point. If we were given an increment of SNR (in dB), the local tradeoff characterizes the relationship between the incremental increase in rate and the reduction of  $P_{\text{out}}$ .

Let us now visualize this local tradeoff by looking at Figure 2-7. When the operating point has  $P_{\text{out}} > \text{SNR}^{-1}$ , we are in the upper-right region, which has a set of parallel lines with slopes 2 and horizontal spacings of 1.5 dB between lines with rate differential 1 b/s/Hz. This means that if we spend all the extra SNR on increasing rate and keep  $P_{\text{out}}$  constant, we can get 2 extra b/s/Hz for every additional 3 dB in SNR. If we spend all the extra SNR on the reduction of  $P_{\text{out}}$  and keep rate constant, we can get 2 orders of magnitude reduction for every additional 10 dB in SNR. We can also get any linear combination of the two extremes because the lines are parallel. Therefore, the local tradeoff is a straight line connecting  $(r, d) = (0, 2)$  and  $(2, 0)$ , which is the lower piece of the *global* tradeoff  $d_{\text{out}}(r)$  in Figure 2-4 extended to  $r = 0$ . Note that the maximum diversity gain of 4 is not achieved.

Similarly, when we operate in the lower-left region,  $P_{\text{out}} < \text{SNR}^{-1}$ , the local tradeoff is a straight line connecting  $(0, 4)$  and  $(4/3, 0)$ . Note that the maximum multiplexing gain of 2 is not achieved.

One key feature in Figure 2-7 is that *the “bending point” moves down*. As rate increases, the outage probability curves do not simply shift right-ward, which is the case for the scalar channel. The larger slopes are achieved at lower  $P_{\text{out}}$  levels.

For system designers, one lesson learned from this local diversity-multiplexing tradeoff study is that depending on the operating point of the system, different segments of the diversity-multiplexing tradeoff curve are important. For two-transmit two-receive antenna systems and target error rate around  $10^{-3}$ , when the operating point is below 30 dB, the  $0 \leq r \leq 1$  segment of the tradeoff is important; above 30 dB, the  $1 \leq r \leq 2$  segment is.

## 2.4 Error Probability and Design Criteria

In this section, we re-derive some pair-wise error probability (PEP) expressions for the multiple antenna channels, and from which, some existing design criteria for good codes are extracted. We also relate the PEP expressions directly to the diversity-multiplexing tradeoff.

The pair-wise error probability can provide a performance lower bound on the overall error probability of a system. When a codeword  $\mathbf{X}_1$  is transmitted, the event of making an error is the union of the events of confusing  $\mathbf{X}_1$  with any of the other codewords,  $\mathbf{X}_2, \mathbf{X}_3, \dots$ . Therefore, by considering the pair with the worst error probability, we obtain a lower bound.<sup>1</sup>

We now evaluate the pair-wise error probability of confusing two codewords  $\mathbf{X}_1$  and  $\mathbf{X}_2$  by first computing the PEP conditioned on a particular channel realization, and then average over all channels according to the Rayleigh distribution.

Let us suppose that there are only two codewords  $\mathbf{X}_1$  and  $\mathbf{X}_2$ ,  $\mathbf{X}_1$  is transmitted, and the realized channel is  $\mathbf{H}$ . In the case of additive white Gaussian noise and maximum likelihood or minimum distance decoding, error happens if the received signal  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$  is closer to  $\mathbf{H}\mathbf{X}_2$  than to  $\mathbf{H}\mathbf{X}_1$ . This happens if the noise magnitude is greater than half of the separation between  $\mathbf{H}\mathbf{X}_1$  and  $\mathbf{H}\mathbf{X}_2$ . Using the well-known approximation of the Gaussian tail function,  $\mathcal{Q}(x) \leq \exp(-x^2/2)$ , the conditional PEP can be approximated by

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2 | \mathbf{H}] \leq \exp \left\{ \frac{(\|\mathbf{H}\mathbf{X}_1 - \mathbf{H}\mathbf{X}_2\|/2)^2}{2\sigma_w^2} \right\} = \exp \left\{ \frac{-\|\mathbf{H}\Delta\|^2}{8\sigma_w^2} \right\}, \quad (2.15)$$

where,  $\Delta = \mathbf{X}_1 - \mathbf{X}_2$ ,  $\sigma_w^2$  is the noise variance per dimension, and  $\|\cdot\|^2$  for a matrix is the total energy of all its entries, also known as the *Frobenius* norm.

Next, we average (2.15) over all channel realizations. Recall that  $\mathbf{H}$  has IID  $\mathcal{CN}(0, 1)$  entries according to the Rayleigh fading assumption. This averaging can be done by moving to the singular value basis of the  $N_t \times T$  matrix  $\Delta$ . We write

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<sup>1</sup>This lower bound is usually good in the high SNR regime when codewords are sufficiently far apart compared to noise levels, so that the nearest neighbor error dominates.

$\Delta = \mathbf{U}\Lambda\mathbf{V}^\dagger$ , where  $\mathbf{U}$  and  $\mathbf{V}$  are unitary matrices, and  $\Lambda$  is a diagonal matrix with the ordered singular values  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{K'} \geq 0$  on its diagonal, where  $K' \stackrel{\text{def}}{=} \min(N_t, T)$ . Now we have,

$$\|\mathbf{H}\Delta\|^2 = \|\mathbf{H}\mathbf{U}\Lambda\mathbf{V}^\dagger\|^2 = \|(\mathbf{H}\mathbf{U})\Lambda\|^2. \quad (2.16)$$

Since  $\mathbf{U}$  is unitary, the entries of  $\Phi \stackrel{\text{def}}{=} \mathbf{H}\mathbf{U}$  are also IID  $\mathcal{CN}(0, 1)$ ,

$$\|\mathbf{H}\Delta\|^2 = \|\Phi\Lambda\|^2 = \sum_{i=1}^{K'} \lambda_i^2 \cdot \sum_{j=1}^{N_r} |\phi_{ji}|^2. \quad (2.17)$$

Therefore,

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq E_{\mathbf{H}} \left[ \exp \left\{ \frac{-\|\mathbf{H}\Delta\|^2}{8\sigma_w^2} \right\} \right] = E_{\Phi} \left[ \exp \left\{ \frac{-\sum_{i=1}^{K'} \lambda_i^2 \cdot \sum_{j=1}^{N_r} |\phi_{ji}|^2}{8\sigma_w^2} \right\} \right]. \quad (2.18)$$

Since  $\phi_{ij}$ 's are independent, we can break up the expectation of products into products of expectations,

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \left( \prod_{i=1}^{K'} E_{\Phi} \left[ \exp \left\{ \frac{-\lambda_i^2 |\phi_{1i}|^2}{8\sigma_w^2} \right\} \right] \right)^{N_r}. \quad (2.19)$$

Each  $|\phi_{ij}|^2$  is a chi-squared random variable with unit variance, averaging over which, we have

$$E_{\Phi} \left[ \exp \left\{ \frac{-\lambda_i^2 |\phi_{1i}|^2}{8\sigma_w^2} \right\} \right] = \frac{1}{1 + \frac{\lambda_i^2}{8\sigma_w^2}}. \quad (2.20)$$

At the end, we obtain the average PEP

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \left( \prod_{i=1}^{K'} \frac{1}{1 + \frac{\lambda_i^2}{8\sigma_w^2}} \right)^{N_r} = \left( \prod_{i=1}^{K'} \left( 1 + \frac{\lambda_i^2}{8\sigma_w^2} \right) \right)^{-N_r}. \quad (2.21)$$

Let us scale the codewords so that the energy per symbol is unity, then  $\frac{\text{SNR}}{N_t} = \frac{1}{\sigma_w^2}$ ,

and we have

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \left( \prod_{i=1}^{K'} \left( 1 + \frac{1}{8N_t} \lambda_i^2 \text{SNR} \right) \right)^{-N_r} \doteq \left( \prod_{i=1}^{K'} (1 + \lambda_i^2 \text{SNR}) \right)^{-N_r}. \quad (2.22)$$

We can ignore the constant  $\frac{1}{8N_t}$  and still keep the exponential growth rate.

From the above average PEP expression, we now derive design criteria that would lead to good codes.

In order to have a good overall performance, we must make sure that there is no particularly bad pair of codewords. Otherwise, a single bad pair could dominate the overall error probability and prevent us from getting good overall performance. Therefore, we want to minimize the quantity

$$\max_{\mathbf{X}_2 \neq \mathbf{X}_1} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \left( \min_{\Delta \neq 0} \prod_{i=1}^{K'} (1 + \lambda_i^2 \text{SNR}) \right)^{-N_r}. \quad (2.23)$$

We see that for each  $\lambda_i = 0$ ,  $1 + \lambda_i^2 \text{SNR} = 1$  for all SNR, and contributes nothing to the total product. When  $\lambda_i > 0$ ,  $1 + \lambda_i^2 \text{SNR}$  behaves like  $\lambda_i^2 \text{SNR}$  at sufficiently high SNR and  $P[\mathbf{X}_1 \rightarrow \mathbf{X}_2]$  decays with SNR. Therefore, the number of effective terms is the number of  $\lambda_i$ 's that are non-zero, i.e., the rank of  $\Delta$ . At sufficiently high SNR,

$$\max_{\mathbf{X}_2 \neq \mathbf{X}_1} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \left( \min_{\Delta \neq 0} \prod_{i=1, \lambda_i \neq 0}^{K'} \lambda_i^2 \text{SNR} \right)^{-N_r} \quad (2.24)$$

$$= \left( \prod_{i=1, \lambda_i \neq 0}^{K'} \lambda_i^2 \right)^{-N_r} \text{SNR}^{-N_r \cdot \min_{\Delta \neq 0} \text{rank}(\Delta)}. \quad (2.25)$$

From the above expression, we obtain three design criteria.

First, the number of terms in the product is  $K' = \min(N_t, T)$ . This suggests that, to have as many effective terms as possible, we want the block code length to be at least  $T \geq N_t$ .

Secondly, the exponent of SNR in (2.25) leads to the rank criterion proved by Tarokh in [35].

**Lemma 2.3** *The Rank Criterion : Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be two distinct codewords, and let  $\Delta = \mathbf{X}_1 - \mathbf{X}_2$  be their difference matrix. If  $\Delta$  has minimum rank  $\kappa$  over the set of any two distinct codewords, then a diversity of  $N_r\kappa$  is achieved.*

Therefore, to design a good codebook that achieves high diversity, we should make sure all difference matrices are full rank. When the first two criteria are met, the maximum diversity of  $N_t N_r$  can be achieved.

The coefficient of SNR in (2.25) gives us the third criterion. When  $\Delta$  is full rank,  $(\prod \lambda_i) = |\det(\Delta)|$ . Therefore, we want to maximize the worst case (smallest) determinant of the difference matrices between all possible pairs of codewords.

The three criteria are summarized here,

1.  $T \geq N_t$ ,
2. all  $\Delta$  should be full rank,
3. maximize the worst case determinant.

Next, we relate the pair-wise error probability expression to diversity-multiplexing tradeoff by writing it as an exponential of SNR.

Let us define  $\text{SNR}^{-\alpha_i} = \lambda_i^2$ , and use  $(x)^+$  to denote  $\max(0, x)$ , as Zheng and Tse did in [41]. We have :

$$1 + \lambda_i^2 \text{SNR} \doteq \text{SNR}^{(1-\alpha_i)^+}, \quad (2.26)$$

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \text{SNR}^{-N_r \sum_{i=1}^{N_t} (1-\alpha_i)^+}, \quad (2.27)$$

$$\max_{\mathbf{X}_2 \neq \mathbf{X}_1} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \text{SNR}^{-N_r \min_{\Delta \neq 0} \sum_{i=1}^{N_t} (1-\alpha_i)^+}. \quad (2.28)$$

Since the worst-case PEP is a lower bound of the overall error probability, the diversity achieved can be upper bounded by  $d \leq N_r \min_{\Delta \neq 0} \sum_{i=1}^{N_t} (1-\alpha_i)^+$ . Later in this thesis, we will use this bound as a means of evaluating diversity-multiplexing tradeoffs achieved by systems. The quantity  $\sum_{i=1}^{N_t} (1-\alpha_i)^+$  is implicitly a function of the multiplexing gain  $r$ . As the rate  $R$  increases with SNR, the codebook and the  $\Delta$  matrices change, which in turn affects the  $\alpha$ 's.



The quantity  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+$  is related to  $\det(\Delta)$ , in the sense that maximizing  $\det(\Delta)$  would lead to large  $\prod (1 + \lambda_i^2 \text{SNR})$ , and then to large  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+$  values.

We note that while (2.28) upper bounds the entire diversity-multiplexing tradeoff curve, (2.25) is related to the diversity gain achieved at  $r = 0$ . When the rate (and the code) is fixed, the coefficient  $\left(\prod_{\lambda_i \neq 0} \lambda_i^2\right)^{-N_r}$  in (2.25) is also fixed, so it grows like  $\text{SNR}^0$ . In this case, the exponent of SNR (negated),  $N_r \min_{\Delta \neq 0} \text{rank}(\Delta)$ , is the diversity gain achieved.<sup>2</sup>

In this section, pair-wise error probability expressions for the multiple antenna channels are derived and related to diversity-multiplexing tradeoff. We can use these formulations to evaluate the performance of a given code. We first identify one bad pair of codewords, use its PEP to lower bound the overall error probability and use the associated  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+$  to upper bound the diversity-multiplexing tradeoff achievable by the system. In the next section, we will apply this technique to evaluate the performance of Gaussian random codes for two-transmit two-receive antenna systems.

## 2.5 Performance of Gaussian Random Codes

Gaussian random codes have often been used by information theorists to study the performance limits of communication systems. In this section, we examine Gaussian random codes of various lengths, and see what diversity-multiplexing tradeoffs can be achieved. To explain why the optimal tradeoff can not be achieved at times, we also look at the tradeoff upper bounds associate with the worst codeword pairs.

It is known that infinitely long Gaussian random codes can achieve the optimal diversity-multiplexing tradeoff. The question of interest here is what tradeoff can be achieved by finite length Gaussian random codes. This would provide valuable benchmarks for more practical finite length codes.

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<sup>2</sup>This is the diversity gain most people referred to before the diversity-multiplexing tradeoff framework was established.

### 2.5.1 Tradeoff Achieved

We now review the diversity-multiplexing tradeoffs achieved by finite length Gaussian random codes evaluated by Zheng and Tse in [41]. They showed that for a system with  $N_t$  transmit and  $N_r$  receive antennas, it is sufficient to have Gaussian random codes with length  $T \geq N_t + N_r - 1$  to achieve the optimal diversity-multiplexing tradeoff in Figure 2-4. Note that it is not necessary to have infinitely long codes if the goal is to achieve only the optimal diversity-multiplexing tradeoff and not the outage probabilities.

However, for shorter Gaussian random codes with  $T < N_t + N_r - 1$ , they showed that the lower bounds on the tradeoffs achieved do not match the optimal tradeoff. They suggested that this could be due to the probability that some codewords getting too close to each other becoming significant for shorter codes.

In the case of two-transmit two-receive antenna systems, Figure 2-9 shows the diversity-multiplexing tradeoff achieved using various Gaussian random codes. When  $T \geq N_t + N_r - 1 = 3$ , optimal tradeoff can be achieved, indicated by the thin solid line.

When  $T = 1$ , we see that the optimal tradeoff is met for  $1 \leq r \leq 2$ , but not for  $0 \leq r < 1$ . Zheng and Tse showed in [41] that this is actually the best tradeoff achievable by any length one code. We can also justify that the optimal tradeoff can not be achieved at  $r = 0$  using the rank criterion stated in Lemma 2.3. It tells us that the maximum diversity achievable by any length one code is  $N_r = 2$ . It is necessary to have  $T \geq N_t = 2$  to achieve the diversity of four.

When  $T = 2$ , a technique called *expurgation* is used to take away codewords that are unnecessarily close. The expurgated Gaussian random codes achieve the tradeoff curve indicated by the dashed line. It achieves the end points, but is sub-optimal for  $0 < r < 1$ . An open question left at the end of their study is whether it is at all possible to achieve the entire optimal tradeoff using length-two codes. We will answer this question later in Chapter 4 by constructing a deterministic length-two code that achieves the optimal diversity-multiplexing tradeoff.

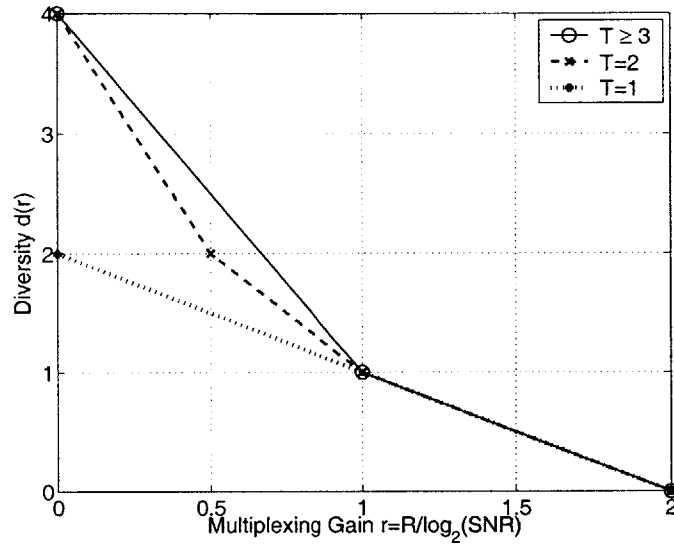


Figure 2-9: Diversity-multiplexing tradeoff achieved using Gaussian random codes of various lengths. Optimal tradeoff is achieved with  $T \geq 3$ .  $T = 2$  codes (with expurgation) can achieve the end points, but is sub-optimal for  $0 < r < 1$ .  $T = 1$  codes only achieve a maximum diversity of  $d = 2$  when  $r = 0$ , which is the most any length one code can do.

## 2.5.2 Worst-Pair Bound

In this section, we illustrate why short Gaussian random codes can not be optimal while the longer ones can. We first identify particularly bad codeword pairs that Gaussian random codebooks are likely to have by using some of the ideas Zheng and Tse developed. We then evaluate the error probabilities associated with these pairs to demonstrate why short Gaussian random codes can not possibly be optimal, and how longer codes avoid this problem.

Gaussian random code matrices have IID  $\mathcal{CN}(0, 1)$  entries. Without loss of generality, let us suppose the first codeword drawn is  $\mathbf{X}_1 = \mathbf{0}$ . If we were to randomly select another codeword  $\mathbf{X}_2$ , their difference is  $\Delta = \mathbf{X}_2$ , with IID  $\mathcal{CN}(0, 1)$  entries<sup>3</sup>. Let us look at the statistics of  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+$  associated with  $\Delta$ , a quantity we introduced in section 2.4. This statistic can help us identify how bad the worst codeword pair is

<sup>3</sup>If  $X_1$  is also random, then  $\Delta$  would have IID  $\mathcal{CN}(0, 2)$  entries. However, the constant factor is not important for diversity-multiplexing tradeoff analysis.

likely to be.

Recall that in the Rayleigh fading model, the channel matrix  $\mathbf{H}$  also has IID  $\mathcal{CN}(0, 1)$  entries, and when we reviewed the outage probability result in Section 2.3, we also looked at the quantity  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+$ . Although the matrix of interest was  $\mathbf{H}$  instead of  $\Delta$ , the statistics is the same. The only difference is that  $\mathbf{H}$  has size  $N_r \times N_t$  and  $\Delta$  has size  $N_t \times T$ .

The outage probability result states that

$$P_{\text{out}}(R, \text{SNR}) \doteq P \left[ \sum_{i=1}^{\min(N_r, N_t)} (1 - \alpha_i)^+ < r \right] \doteq \text{SNR}^{-d_{\text{out}}(r)}, \quad (2.29)$$

where the optimal tradeoff  $d_{\text{out}}(r)$  is defined in Lemma 2.2 and plotted in Figure 2-4.

To obtain the statistics of  $\Delta$ , we replace  $r$  with  $d_{\text{out}}^{-1}(rT)$  and obtain

$$P \left[ \sum_{i=1}^{\min(N_t, T)} (1 - \alpha_i)^+ < d_{\text{out}}^{-1}(rT) \right] \doteq \text{SNR}^{-rT}, \quad (2.30)$$

i.e., if we were to choose a codeword  $\mathbf{X}_2$  randomly, the probability of the resulting quantity  $\sum (1 - \alpha_i)^+$  being less than  $d_{\text{out}}^{-1}(rT)$  would be about  $\text{SNR}^{-rT}$ . There are  $\text{SNR}^{rT}$  codewords in a codebook. Therefore, the probability that one of them having  $\sum (1 - \alpha_i)^+ < d_{\text{out}}^{-1}(rT)$  is order 1. This means that for a Gaussian random codebook, there is a very high probability that there are codeword pairs with

$$\sum_{i=1}^{\min(N_t, T)} (1 - \alpha_i)^+ < d_{\text{out}}^{-1}(rT). \quad (2.31)$$

These would be the particularly bad codeword pairs that could dominate the overall error probability. Even if we were willing to re-select codewords when the realized codewords are particularly bad, i.e., expurgate bad codewords, we would still only be able to guarantee the worst  $\sum (1 - \alpha_i)^+$  to be about the same as  $d_{\text{out}}^{-1}(rT)$ . Anything better would be impossible to get selected randomly.

Next, we evaluate the error probability associated with this bad codeword pair

with  $\sum_{i=1}^{N_t} (1 - \alpha_i)^+ \approx d_{\text{out}}^{-1}(rT)$ . From (2.27), we have

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \text{SNR}^{-N_r \sum_{i=1}^{N_t} (1 - \alpha_i)^+} \doteq \text{SNR}^{-N_r d_{\text{out}}^{-1}(rT)}. \quad (2.32)$$

The overall error probability is lower bounded by this worst-pair error probability, so the diversity multiplexing tradeoff achieved is at most

$$d(r) < N_r d_{\text{out}}^{-1}(rT). \quad (2.33)$$

For the particular case of two-transmit two-receive antenna systems with code length  $T$ , this upper bound becomes  $d(r) < 2d_{\text{out}}^{-1}(rT)$ , where  $d_{\text{out}}(r)$  is a piece-wise linear curve connecting the points  $(0, 2T)$ ,  $(1, T - 1)$ , and  $(2, 0)$ , computed from Lemma 2.1 with the parameters  $N_r = 2$  and  $N_t = T$ . With simple manipulation, we determine that the upper bound  $d(r) < 2d_{\text{out}}^{-1}(rT)$  is a piece-wise linear curve connecting the points  $(0, 4)$ ,  $(\frac{T-1}{T}, 2)$ , and  $(2, 0)$ . These curves evaluated for  $T = 1, 2, 3, 4$  and  $\infty$  are plotted as thin solid lines in Figure 2-10, together with the optimal tradeoff curve shown as a thicker dashed line.

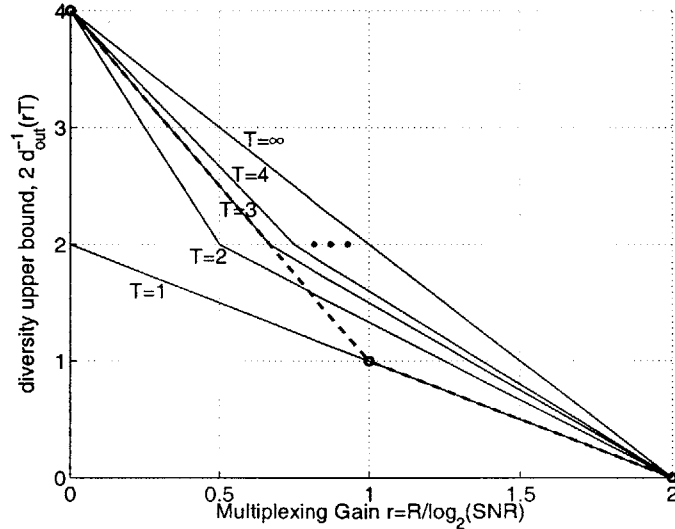


Figure 2-10: The upper bound on the diversity-multiplexing tradeoff achievable using Gaussian random codes based on the worst-pair error probability,  $d(r) < 2d_{\text{out}}^{-1}(rT)$ . Short codes with  $T \leq 2$  are sub-optimal due to the worst-pair being particularly bad.

We can see that for  $T = 1$  and  $T = 2$ , the upper bound curves are below the optimal tradeoff curve. This is because by choosing codebooks randomly, we are unable to avoid getting particularly bad codeword pairs, which in turn prevents us from reaching the optimal tradeoff. As  $T$  increases, it becomes less likely for us to get bad codeword pairs and the upper bound rises.

Let us compare Figure 2-10 and Figure 2-9. When the corresponding curves agree, it means that the performance of the Gaussian random code is fully justifiable using the single worst codeword pairs. An example is the  $0 \leq r \leq 0.5$  segment of  $T = 2$ . When they do not agree, it means there are multiple bad codeword pairs. An example is the entire  $T = \infty$  curve.

In summary, we saw in this section that for a two-transmit two-receive antenna system, Gaussian random codes with  $T \geq 3$  can reach optimal diversity-multiplexing tradeoff and shorter codes with  $T \leq 2$  are sub-optimal due to particularly bad codeword pairs. While we can not do better in the  $T = 1$  case, there is still room for improvement in the  $T = 2$  case.

In the next three chapters, we will study coding and decoding strategies for the two-transmit two-receive antenna channel, at three different code lengths,  $T = 1$ ,  $T = 2$ , and large  $T$ . We will design and analyze practical deterministic codes instead of using random ones.

# Chapter 3

## Uncoded Systems and Efficient Detection

### 3.1 Introduction

In this chapter, we study the case of communication using multiple antennas with no coding involved. This is the simplest system and incurs no delay. We will study more complex systems with various degrees of coding and delay in the later chapters.

More specifically, we restrict the transmitted signal to be a vector  $\mathbf{x}$  with entries drawn independently from some QAM-like constellation. We look at the problem of detecting  $\mathbf{x}$  from the received signal  $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$ , where the  $N_r \times N_t$  channel matrix  $\mathbf{H}$  is known at the receiver but not at the transmitter, and  $\mathbf{w}$  is the additive white Gaussian noise vector.

The key problem here is the interference between the entries of  $\mathbf{x}$ . When  $\mathbf{x}$  is multiplied by  $\mathbf{H}$ , its entries are linearly combined. This interference makes the detection problem at the receiver difficult.

For a system designer, the goal is to handle the interference with good complexity-performance tradeoff. At one end of the spectrum, maximum likelihood detection (MLD) is optimal, but its complexity generally makes it impractical. A variety of other detectors, both linear and nonlinear, require substantially less complexity, but sacrifice significant amount of performance.

In this chapter, we present lattice reduction (LR) techniques and use them in conjunction with traditional low-complexity linear and nonlinear detectors to substantially close their gaps to the fundamental performance limits with little additional system complexity.

For most of this chapter, we focus on the two-transmit two-receive antenna case. The technique introduced can be extended to higher dimensions. However, the complexity increases. We comment on this at the end of the chapter.

For the LR based detection techniques proposed, we evaluate the complexity and performance for both Gaussian channel (fixed  $\mathbf{H}$ ) and Rayleigh fading channel (random  $\mathbf{H}$ ) cases. We show that, relative to the maximum likelihood bound, LR techniques get us to within 3 dB for any Gaussian channel, and allow us to achieve the same diversity on the Rayleigh fading channel when sufficiently large constellations are used. We also show that, in the fading case, systems with uncoded transmission together with LR based detection can effectively achieve the optimal diversity-multiplexing tradeoff achievable by any length-one code.

This chapter is outlined as follows. We first review some traditional detectors and discuss their respective problems. We then look at the various detectors graphically, which leads to the idea of operating in a reduced lattice basis. We identify the optimal basis and present an iterative algorithm for obtaining it. We then evaluate the complexity and performance for Gaussian and Rayleigh fading channel cases. Finally, we discuss the dual problem of applying lattice reduction to pre-coding at the transmitter, as well as how the LR idea can be extended to higher dimensions.

## 3.2 Traditional Detectors

In this section, we briefly review three traditional detectors and compare them graphically, which will lead to the lattice reduction idea.

An important performance bound corresponds to the *maximum likelihood detection*, which minimizes the probability of block error. In the case where the noise is



AWGN, the minimum distance rule is used,

$$\hat{\mathbf{x}}_{\text{MLD}} = \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2. \quad (3.1)$$

In the absence of special structure, MLD requires computing distances to every code-word to find the closest one. Therefore, it has exponential complexity in transmission rate.

By contrast, linear detectors have much lower complexity. They take the form of  $\hat{\mathbf{x}} = f(\mathbf{A}\mathbf{y})$ , where  $\mathbf{A}$  is some matrix and  $f(\cdot)$  is a slicer, which quantizes each entry of  $\mathbf{A}\mathbf{y}$  to the nearest constellation symbol to obtain  $\hat{\mathbf{x}}$ . For familiar constellations such as 4-QAM or 16-QAM, this quantization can be implemented with very little complexity.

The choice  $\mathbf{A} = \mathbf{H}^{-1}$  corresponds to what is sometimes referred to as *inverse channel detection* (ICD) [29], or in the case of the multiuser detection problem, the decorrelator.<sup>1</sup> As is well-known, the performance of ICD can suffer dramatically due to noise enhancement if  $\mathbf{H}$  is near singular. Indeed, since  $\mathbf{H}^{-1}\mathbf{y} = \mathbf{x} + \mathbf{H}^{-1}\mathbf{w}$ , the effective noise at the slicer input is  $\mathbf{H}^{-1}\mathbf{w}$ . Other linear detectors include the minimum mean square error (MMSE) detector, which offers slightly better performance by mitigating noise enhancement, but is still far from the performance of MLD.

A class of nonlinear detectors that offer better performance with only a modest increase in complexity is that based on successive cancellation. An example is the *Bell Labs Layered Space-Time* (BLAST) receiver [9]. The basic steps of the simplest version of BLAST detection are nulling and cancellation.

*Nulling* : First, the channel matrix is factored as  $\mathbf{H} = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q}$  is unitary and  $\mathbf{R}$  is upper triangular. Next, the received signal is pre-processed to obtain  $\mathbf{y}' = \mathbf{Q}^\dagger \mathbf{y} = \mathbf{R}\mathbf{x} + \mathbf{w}'$ , where  $\mathbf{w}' = \mathbf{Q}^\dagger \mathbf{w}$ , with  $\dagger$  denoting the conjugate transpose

---

<sup>1</sup> $\mathbf{H}^{-1}$  is replaced by the pseudo-inverse of  $\mathbf{H}$  if it is not square.

operation, so that

$$\begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{N_t} \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1N_t} \\ 0 & r_{22} & \cdots & r_{2N_t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{N_t N_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_t} \end{bmatrix} + \begin{bmatrix} w'_1 \\ w'_2 \\ \vdots \\ w'_{N_t} \end{bmatrix}. \quad (3.2)$$

*Cancellation* : Using the pre-processed data (3.2), the entries of  $\mathbf{x}$  are detected one by one in decreasing order. Specifically, after detecting  $x_k, \dots, x_{N_t}$ , we can subtract their interference out of  $y'_{k-1}$  to detect  $x_{k-1}$ .

If each  $\hat{x}_k$  were not quantized to the nearest constellation symbol as we proceeded, this form of detection would specialize to ICD. Thus, this quantization serves an important noise-cancellation role.

A major problem with BLAST detection is error propagation. The entry detected first usually has the smallest signal to noise ratio and the most error. Unfortunately, detecting later entries correctly vitally depends on having correctly decoded previous entries. For this reason, in an uncoded system, where error correction is not used, the error rate for BLAST detection is typically dominated by that of the first entry, and therefore, far from optimal.

To develop a framework within which to introduce lattice reduction, we consider MLD, ICD, and BLAST detection in the  $2 \times 2$  (real) example shown in Figure 3-1. The transmitted symbols  $x_1$  and  $x_2$  are each integers within a large range, and the channel matrix is, for purpose of illustration,  $\mathbf{H} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ .

The received constellation  $\mathbf{H}\mathbf{x}$  is shown in (a). It can be viewed as a lattice with basis vectors being the two columns of  $\mathbf{H}$ , which are drawn to show the distortion of the lattice. The decision boundaries for ICD, BLAST detection, and MLD are shown in (b), (c), and (d), respectively. For ICD, the decision regions are undesirably elongated and narrow parallelograms; points far away are undesirably included and the minimum amount of noise needed for an error to occur, which is the size of the inscribed circle drawn, is small. This is due to the two basis vectors being highly

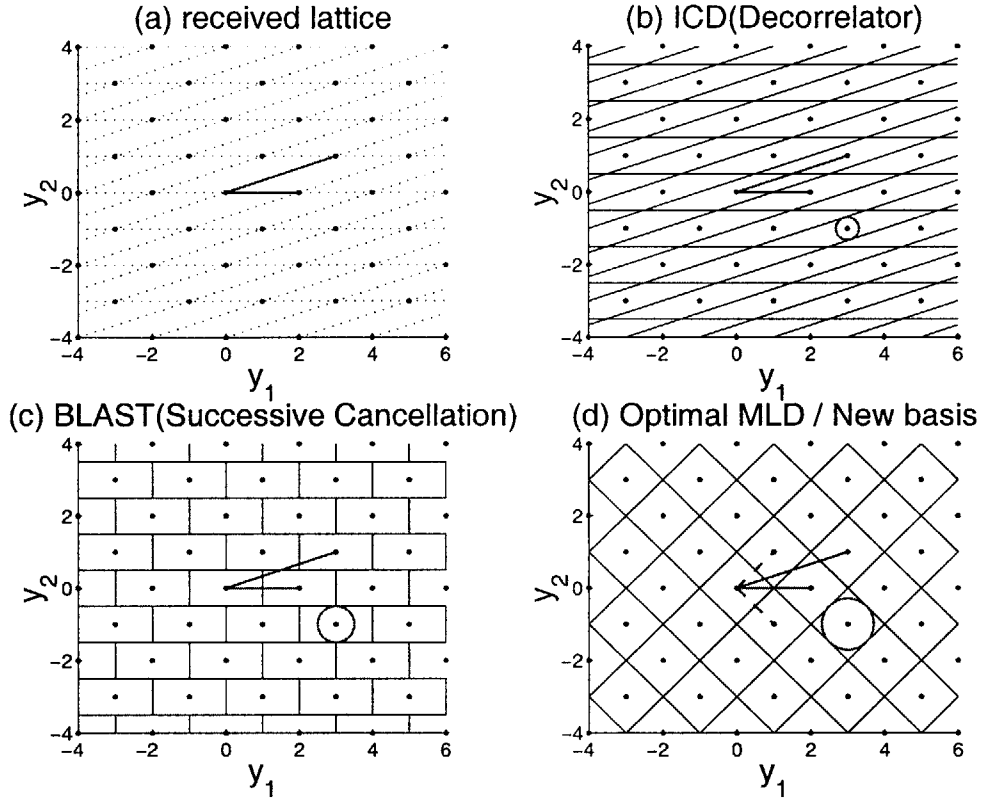


Figure 3-1: Comparison of decision boundaries for various detection methods.

correlated. For BLAST, the decision regions are rectangular, because one entry of  $\mathbf{x}$  is detected at a time. While better than ICD, it is still inferior to the optimal decision boundary drawn in (d), whose optimality is apparent by inspection.

In this particular example, if we were to consider the lattice basis vectors to be  $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$  and  $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$  instead of  $\begin{bmatrix} 2 & 0 \end{bmatrix}^T$  and  $\begin{bmatrix} 3 & 1 \end{bmatrix}^T$ , where  $^T$  denotes the transpose operator, then the decision boundaries for ICD and BLAST detection would coincide with those of MLD, and therefore be optimal.

While a basis change cannot always lead to optimum performance, it can in general improve performance. In particular, changing the lattice basis to be more orthogonal and shorter, the sense of which we will make precise later, we can generally obtain better decision boundaries. The more correlated the columns of  $\mathbf{H}$ , the more significant the improvements. Note that changing lattice basis does not change the lattice,

so the underlying detection problem remains the same. The problem of finding the optimal lattice basis is called the *lattice reduction* (LR) problem.

Our goal is to use lattice reduction to help us find the lattice point nearest to the received signal point. This problem is more generally known as the *lattice decoding* problem. It has been studied for the case of AWGN channel. In that case, there is no channel distortion. The lattice is freely designed instead of imposed by the channel, so very efficient algorithms can be designed for decoding a highly structured lattice, for example, the Leech lattice [6]. In our case, we must consider decoding for general lattices, and complexity is of great concern.

### 3.3 Lattice Reduction

A lattice in  $n$  complex dimensions can be described by

$$\mathcal{L} = \{\mathbf{s} \mid \mathbf{s} = \mathbf{B}\boldsymbol{\lambda}\}, \quad (3.3)$$

where  $\mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n]$  is a matrix whose columns are basis vectors for the lattice and  $\boldsymbol{\lambda} = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_n]^T$  is a vector of complex integer weights, i.e.,  $\lambda_i \in \mathbb{Z} + \mathbb{Z}j$  with  $\mathbb{Z}$  denoting the set of integers.

For any lattice  $\mathcal{L}$  there are many possible bases. Indeed, if  $\mathbf{B}$  is a basis, so is  $\mathbf{B}' = \mathbf{B}\mathbf{P}$  for any matrix  $\mathbf{P}$  such that both  $\mathbf{P}$  and  $\mathbf{P}^{-1}$  have integer entries. Specifically, a point  $\mathbf{s}$  represented by  $\mathbf{x}$  in the basis  $\mathbf{B}$  is represented by  $\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$  in the basis  $\mathbf{B}'$ , i.e.,  $\mathbf{s} = \mathbf{B}\mathbf{x} = (\mathbf{B}\mathbf{P})(\mathbf{P}^{-1}\mathbf{x}) = \mathbf{B}'\mathbf{z}$ .

The basic idea behind using lattice reduction in conjunction with traditional low-complexity detectors is to operate in a chosen lattice basis that is optimized for those detectors, as shown in Figure 3-2.

In the traditional system, the detector compensates for the original channel  $\mathbf{H}$  to produce  $\hat{\mathbf{x}}$ . In the new system, we perform a basis change via a matrix  $\mathbf{P}$ , specifically

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} = (\mathbf{H}\mathbf{P})(\mathbf{P}^{-1}\mathbf{x}) + \mathbf{w} = \mathbf{H}'\mathbf{z} + \mathbf{w}. \quad (3.4)$$

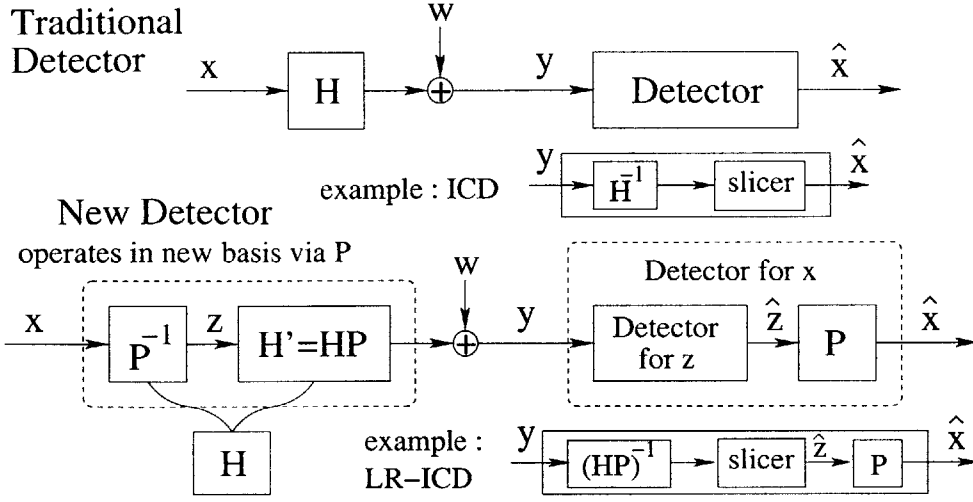


Figure 3-2: Using lattice reduction in conjunction with traditional detectors.

With this basis change, the traditional detector is first used to compensate for the new channel  $\mathbf{H}' = \mathbf{H}\mathbf{P}$  to produce  $\hat{\mathbf{z}}$ , then produce  $\hat{\mathbf{x}}$  via  $\hat{\mathbf{x}} = \mathbf{P}\hat{\mathbf{z}}$ . For example, if ICD is employed, then  $(\mathbf{H}')^{-1}\mathbf{y}$  is quantized to produce  $\hat{\mathbf{z}}$ , from which we obtain  $\hat{\mathbf{x}}$  via  $\hat{\mathbf{x}} = \mathbf{P}\hat{\mathbf{z}}$ .

### 3.3.1 Choice of optimal basis

Let us now discuss what new basis is optimal to operate in. First, we note that ICD and BLAST detection are more effective when the channel matrix is further from being singular. Geometrically, this corresponds to wanting the columns of the new  $\mathbf{H}'$ , which are the new basis vectors of the received constellation lattice, to be less correlated and shorter. Thus, the problem of improving the condition of  $\mathbf{H}'$  is one of reducing the lattice basis corresponding to  $\mathbf{H}$ .

In the  $2 \times 2$  case,  $\mathbf{H} = [\mathbf{b}_1 \ \mathbf{b}_2]$ . Let us use “ $\sim$ ” to denote the component of one basis vector that is orthogonal to the other one. In particular,  $\tilde{\mathbf{b}}_1$  denotes the component of  $\mathbf{b}_1$  that is orthogonal to  $\mathbf{b}_2$ , and  $\tilde{\mathbf{b}}_2$  is similarly defined. For BLAST detection, the effective SNR at the point of detecting  $x_1$  and  $x_2$  are  $r_{11} = \|\mathbf{b}_1\|$  and  $r_{22} = \|\tilde{\mathbf{b}}_2\|$ , respectively. Therefore, the best basis is the one with the largest  $\min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|)$ . For ICD, the corresponding measure is  $\min(\|\tilde{\mathbf{b}}_1\|, \|\tilde{\mathbf{b}}_2\|)$ , which evaluates to  $\|\tilde{\mathbf{b}}_1\|$  when

$\|\mathbf{b}_1\| \leq \|b_2\|$ . With these criteria, we show that the optimal basis for both detection methods is  $(\mathbf{u}, \mathbf{v})$ , where  $\mathbf{u}$  is the shortest (non-zero) vector in the lattice and  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ . This is done in the next two lemmas, for BLAST detection and ICD, respectively.

**Lemma 3.1** (*Optimality for BLAST*) *Given a two dimensional lattice with basis  $(\mathbf{u}, \mathbf{v})$ . If  $\mathbf{u}$  is the shortest (non-zero) vector in the lattice and  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ , then for any other basis of the lattice  $(\mathbf{b}_1, \mathbf{b}_2)$ ,*

$$\min(\|\mathbf{u}\|, \|\tilde{\mathbf{v}}\|) \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|). \quad (3.5)$$

*Proof:*

This proof can be done in two parts:

$$1) \|\mathbf{u}\| \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|) \text{ and } 2) \|\tilde{\mathbf{v}}\| \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|).$$

1) Since  $(\mathbf{b}_1, \mathbf{b}_2)$  is a lattice basis,  $\mathbf{u}$  can be written as  $\mathbf{u} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ , where  $c_1$  and  $c_2$  are not both zero and  $c_1, c_2 \in \mathbb{Z} + \mathbb{Z}j$ .

If  $c_2 \neq 0$ , then  $|c_2| \geq 1$ . The component of  $\mathbf{u}$  orthogonal to  $\mathbf{b}_1$  is  $c_2\tilde{\mathbf{b}}_2$ . Therefore,

$$\|\mathbf{u}\| \geq \|c_2\tilde{\mathbf{b}}_2\| \geq \|\tilde{\mathbf{b}}_2\| \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|).$$

If  $c_2 = 0$ , then  $\mathbf{u} = c_1\mathbf{b}_1$ , where  $c_1 \neq 0$  and  $|c_1| \geq 1$ . Therefore,

$$\left. \begin{array}{l} \|\mathbf{u}\| = |c_1| \cdot \|\mathbf{b}_1\| \geq 1 \cdot \|\mathbf{b}_1\| \\ \mathbf{u} \text{ shortest} \Rightarrow \|\mathbf{u}\| \leq \|\mathbf{b}_1\| \end{array} \right\} \Rightarrow \|\mathbf{u}\| = \|\mathbf{b}_1\| \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|).$$

2) Since  $(\mathbf{u}, \mathbf{v})$  and  $(\mathbf{b}_1, \mathbf{b}_2)$  are both bases of the same lattice,  $\|\mathbf{u}\| \cdot \|\tilde{\mathbf{v}}\| = \|\mathbf{b}_1\| \cdot \|\tilde{\mathbf{b}}_2\|$ , both being volume of a unit cell of the lattice. Therefore,

$$\mathbf{u} \text{ shortest} \Rightarrow \|\mathbf{u}\| \leq \|\mathbf{b}_1\| \Rightarrow \|\tilde{\mathbf{v}}\| \geq \|\tilde{\mathbf{b}}_2\| \geq \min(\|\mathbf{b}_1\|, \|\tilde{\mathbf{b}}_2\|).$$

■

**Lemma 3.2** (*Optimality for ICD*) *Given a two dimensional lattice with basis  $(\mathbf{u}, \mathbf{v})$ . If  $\mathbf{u}$  is the shortest (non-zero) vector in the lattice and  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ , then for any other basis of the lattice  $(\mathbf{b}_1, \mathbf{b}_2)$  with  $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ ,*

$$\|\tilde{\mathbf{u}}\| \geq \|\tilde{\mathbf{b}}_1\|. \quad (3.6)$$

*Proof:*

This proof is done by contradiction. Suppose  $\|\tilde{\mathbf{u}}\| < \|\tilde{\mathbf{b}}_1\|$ . Since  $(\mathbf{u}, \mathbf{v})$  and  $(\mathbf{b}_1, \mathbf{b}_2)$  are both bases of the same lattice,  $\|\tilde{\mathbf{u}}\| \cdot \|\mathbf{v}\| = \|\tilde{\mathbf{b}}_1\| \cdot \|\mathbf{b}_2\|$ , again, both being volume of a unit cell of the lattice.

$$\begin{aligned}
\|\tilde{\mathbf{u}}\| < \|\tilde{\mathbf{b}}_1\| &\implies \|\mathbf{v}\| > \|\mathbf{b}_2\| \geq \|\mathbf{b}_1\| \\
&\implies \text{Both } \mathbf{b}_1 \text{ and } \mathbf{b}_2 \text{ are multiples of } \mathbf{u}. \\
&\implies (\mathbf{b}_1, \mathbf{b}_2) \text{ can not be a basis.} \\
&\implies \text{contradiction}
\end{aligned}$$

This utilizes the condition that  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ . Therefore,  $\mathbf{b}_1$  and  $\mathbf{b}_2$  can not both be shorter than  $\mathbf{v}$  and form a basis. ■

### 3.3.2 Reduction Algorithm

Given an original set of basis vectors  $(\mathbf{b}_1, \mathbf{b}_2)$  for a lattice with  $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ , we develop an iterative algorithm to progressively reduce their correlation and converge to the desired basis vectors  $(\mathbf{u}, \mathbf{v})$ .

One intuitive way to reduce the correlation between two lattice basis vectors is to subtract integer copies of the shorter vector out of the longer one. Let  $\mathbf{b}'_2 = (\mathbf{b}_2 - n\mathbf{b}_1)$  be the replacement for  $\mathbf{b}_2$ . The parameter  $n$  should be chosen so as to minimize the correlation between  $\mathbf{b}_1$  and  $\mathbf{b}'_2$ , i.e.,

$$n^* = \arg \min_{n \in \mathbb{Z} + \mathbb{Z}j} |\langle \mathbf{b}_1, \mathbf{b}_2 - n\mathbf{b}_1 \rangle| = \arg \min_{n \in \mathbb{Z} + \mathbb{Z}j} |\langle \mathbf{b}_1, \mathbf{b}_2 \rangle - n\|\mathbf{b}_1\|^2| = \left\lfloor \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} \right\rfloor, \quad (3.7)$$

where the function  $\lfloor \cdot \rfloor$  rounds its argument to the nearest integer. For complex arguments, real and imaginary parts are rounded separately. And to avoid ambiguity, half integers are rounded to even integers. Note that this choice of  $n$  given by (3.7)

also minimizes the norm of  $\mathbf{b}'_2$ .

$$\begin{aligned}
& \arg \min_{n \in \mathbb{Z} + \mathbb{Z}j} \|\mathbf{b}_2 - n\mathbf{b}_1\|^2 \\
&= \arg \min_{n \in \mathbb{Z} + \mathbb{Z}j} |n|^2 \|\mathbf{b}_1\|^2 - 2 \operatorname{Re}\{n \langle \mathbf{b}_2, \mathbf{b}_1 \rangle\} + \|\mathbf{b}_2\|^2 \\
&= \arg \min_{n=n_r+n_i j} (n_r^2 + n_i^2) \|\mathbf{b}_1\|^2 - 2n_r \operatorname{Re}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\} - 2n_i \operatorname{Im}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\} + \|\mathbf{b}_2\|^2 \\
&= \left\lfloor \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} \right\rfloor.
\end{aligned}$$

The resulting correlation after replacing  $\mathbf{b}_2$  with  $\mathbf{b}'_2$  is

$$\begin{aligned}
\langle \mathbf{b}_1, \mathbf{b}'_2 \rangle &= \left\langle \mathbf{b}_1, \left( \mathbf{b}_2 - \left\lfloor \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} \right\rfloor \mathbf{b}_1 \right) \right\rangle \\
&= \left( \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} - \left\lfloor \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} \right\rfloor \right) \cdot \|\mathbf{b}_1\|^2.
\end{aligned}$$

Since the rounding errors for real and imaginary parts are each no more than  $1/2$ , we have

$$|\operatorname{Re}\{\langle \mathbf{b}_1, \mathbf{b}'_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2 \text{ and } |\operatorname{Im}\{\langle \mathbf{b}_1, \mathbf{b}'_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2. \quad (3.8)$$

After replacing  $\mathbf{b}_2$  with the optimal  $\mathbf{b}'_2$ , if this new  $\mathbf{b}_2$  is shorter than  $\mathbf{b}_1$ , we swap them and then check whether further subtraction is possible.

Summarizing, the algorithm is as follows:

1. Check the correlation. If  $|\operatorname{Re}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2$  and  $|\operatorname{Im}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2$ , stop. Otherwise, replace  $\mathbf{b}_2$  with  $\mathbf{b}_2 - \left\lfloor \frac{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle}{\|\mathbf{b}_1\|^2} \right\rfloor \mathbf{b}_1$  and go to step 2.
2. Check their lengths. If  $\|\mathbf{b}_2\| > \|\mathbf{b}_1\|$ , stop. Otherwise, swap them and go to step 1.

When this iterative procedure stops, the resulting basis has the properties  $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$ ,  $|\operatorname{Re}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2$  and  $|\operatorname{Im}\{\langle \mathbf{b}_1, \mathbf{b}_2 \rangle\}| \leq \frac{1}{2} \|\mathbf{b}_1\|^2$ . It follows that basis vectors with these properties are the ones we desire, as we show next.



**Lemma 3.3** *Given a two dimensional lattice with basis vectors  $\mathbf{u}$  and  $\mathbf{v}$ . If  $\|\mathbf{u}\| \leq \|\mathbf{v}\|$ ,  $|\operatorname{Re}\{\langle \mathbf{u}, \mathbf{v} \rangle\}| \leq \frac{1}{2}\|\mathbf{u}\|^2$ , and  $|\operatorname{Im}\{\langle \mathbf{u}, \mathbf{v} \rangle\}| \leq \frac{1}{2}\|\mathbf{u}\|^2$ , then*

- 1)  $\mathbf{u}$  is the shortest (non-zero) vector in the lattice.
- 2)  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ .

*Proof:*

1) Since  $(\mathbf{u}, \mathbf{v})$  is a lattice basis, any vector  $\mathbf{s}$  in the lattice can be written as  $\mathbf{s} = a\mathbf{u} + b\mathbf{v}$ , with  $a, b \in \mathbb{Z} + \mathbb{Z}j$ . Let  $a_r = \operatorname{Re}\{a\}$ ,  $a_i = \operatorname{Im}\{a\}$ ,  $b_r = \operatorname{Re}\{b\}$ ,  $b_i = \operatorname{Im}\{b\}$ . We have,

$$\begin{aligned} \|\mathbf{s}\|^2 &= \|a\mathbf{u} + b\mathbf{v}\|^2 \\ &= |a|^2\|\mathbf{u}\|^2 + |b|^2\|\mathbf{v}\|^2 + 2\operatorname{Re}\{a^\dagger b\langle \mathbf{u}, \mathbf{v} \rangle\} \\ &\geq (a_r^2 + a_i^2 + b_r^2 + b_i^2 - |a_r b_r + a_i b_i| - |a_i b_r - a_r b_i|)\|\mathbf{u}\|^2 \\ &\geq \|\mathbf{u}\|^2 \quad \text{when } a_r, a_i, b_r, b_i \text{ are not all 0,} \end{aligned}$$

The last step uses the identities, for  $a, b, c, d \in \mathbb{Z}$ ,

- $a^2 + b^2 + c^2 + d^2 \geq |ac| + |bd| + |bc| + |ad|$  with equality iff  $|a| = |b| = |c| = |d|$ .
- $|ac| + |bd| \geq |ac + bd|$  with equality iff  $abcd \geq 0$ .
- $|bc| + |ad| \geq |bc - ad|$  with equality iff  $abcd \leq 0$ .

2) Any vector  $\mathbf{s}$  in the lattice that is not a multiple of  $\mathbf{u}$  can be written as  $\mathbf{s} = a\mathbf{u} + b\mathbf{v}$ ,  $a, b \in \mathbb{Z} + \mathbb{Z}j$ , and  $b \neq 0$ .

$$\begin{aligned} \|\mathbf{s}\|^2 &= \|a\mathbf{u} + b\mathbf{v}\|^2 \\ &= |b|^2 (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2) + \underbrace{|a|^2\|\mathbf{u}\|^2 + |b|^2\|\mathbf{u}\|^2 + 2\operatorname{Re}\{a^\dagger b\langle \mathbf{u}, \mathbf{v} \rangle\}}_{+ \|\mathbf{u}\|^2} + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2) \\ &\geq |b|^2 (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2) + \|\mathbf{u}\|^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2) \\ &= (|b|^2 - 1) \cdot (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2) + \|\mathbf{v}\|^2 \\ &\geq \|\mathbf{v}\|^2 \quad \text{because } b \neq 0 \end{aligned}$$

■

### 3.3.3 Convergence and Complexity

In this section, let us discuss the convergence of the iterative algorithm proposed as well as its complexity. In other words, does this procedure end and how many iterations does it take?

It is clear that the procedure does end. In particular, after each iteration, the lengths of both basis vectors decrease (at least one decreases strictly); otherwise,

the procedure ends. Since lattices are discrete, there can be only a finite number of vectors shorter than the original ones. Thus, the procedure must end.

Showing that the algorithm converges is not enough. Even if it converges, it could still take many iterations to finish. To get some intuition on the number of iterations needed, let us look at the  $2 \times 2$  real case instead of the complex case. There are relatively fewer parameters which makes it easier to study.

For the two dimensional real case,  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are each described by 2 real numbers. However, the number of iterations needed is only a function of the relative angle between  $\mathbf{b}_1$  and  $\mathbf{b}_2$  and the ratio of their lengths. Rotating and scaling the vectors together do not matter. Therefore, without loss of generality, we can fix  $\mathbf{b}_1$  to be  $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ .

To help us gain an overall understanding of all the possibilities, Figure 3-3 shows the number of iterations needed for values of  $\mathbf{b}_2(1)$  and  $\mathbf{b}_2(2)$  ranging from 0 to 1 in 0.01 increments.<sup>2</sup>

From Figure 3-3, we see that in most cases, the procedure finishes within two iterations. In order to have a large number of iterations,  $\mathbf{b}_2$  has to take on very special values. We notice that there is a fractal look to this figure. This motivates us to look for special examples that requires large numbers of iterations to reduce. A special example related to the Fibonacci numbers is found.

The Fibonacci number series is defined by,  $F_1 = 1$ ,  $F_2 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$ . If we continue expending the terms, we get

$$F_n = F_{n-1} + F_{n-2} = 2F_{n-2} + F_{n-3} = 3F_{n-2} - F_{n-4} \implies \left\lfloor \frac{F_n}{F_{n-2}} \right\rfloor = 3. \quad (3.9)$$

The special example we construct is  $\mathbf{b}_1 = \begin{bmatrix} F_{n-2} & 0 \end{bmatrix}^T$  and  $\mathbf{b}_2 = \begin{bmatrix} F_n & \epsilon_n \end{bmatrix}^T$ , where  $n$  is arbitrarily large and  $\epsilon_n$  is sufficiently small so that the second entry does not affect the iterations. We need  $\epsilon_n \neq 0$  so that  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are linearly independent.

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<sup>2</sup>This region is chosen because the four quadrants of  $(\mathbf{b}_2(1), \mathbf{b}_2(2))$  have symmetry, so focusing on the first quadrant is sufficient. Also, if  $\mathbf{b}_2$  starts outside of this region, it will come into it after one iteration (when  $|\mathbf{b}_2(2)| \leq 1$ ), or stop within one iteration (when  $|\mathbf{b}_2(2)| > 1$ ).

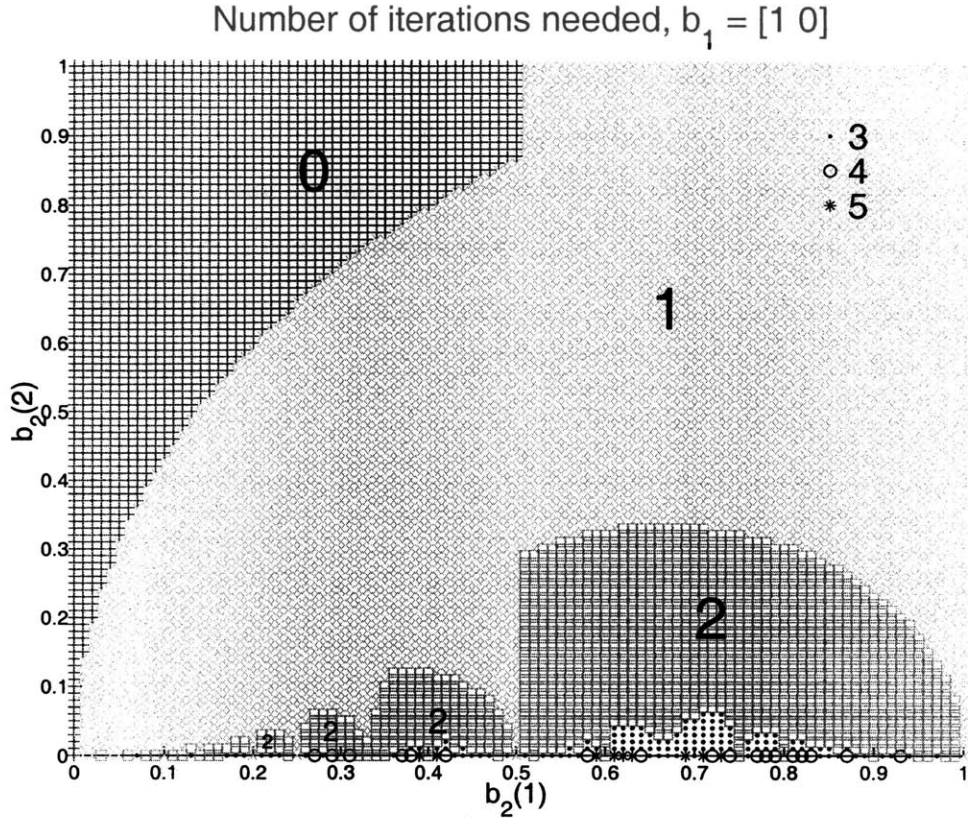


Figure 3-3: Number of iterations needed to find the optimal reduced basis.  $\mathbf{b}_1$  is fixed at  $[1 \ 0]^T$ , each entry of  $\mathbf{b}_2$  ranges from 0 to 1 in 0.01 increments.

(Note that in Figure 3-3, all cases that require large numbers of iterations occur near the horizontal axis.) The reduction procedure happen as follows.

$$\begin{aligned}
 \mathbf{b}'_2 &= \mathbf{b}_2 - 3\mathbf{b}_1 = \begin{bmatrix} -F_{n-4} \\ \sim \end{bmatrix}, \quad \text{SWAP,} \quad \longrightarrow \quad \mathbf{b}_1 = \begin{bmatrix} -F_{n-4} \\ \sim \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} F_{n-2} \\ \sim \end{bmatrix}, \\
 \mathbf{b}'_2 &= \mathbf{b}_2 + 3\mathbf{b}_1 = \begin{bmatrix} -F_{n-6} \\ \sim \end{bmatrix}, \quad \text{SWAP,} \quad \longrightarrow \quad \mathbf{b}_1 = \begin{bmatrix} -F_{n-6} \\ \sim \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -F_{n-4} \\ \sim \end{bmatrix}, \\
 \mathbf{b}'_2 &= \mathbf{b}_2 - 3\mathbf{b}_1 = \begin{bmatrix} F_{n-8} \\ \sim \end{bmatrix}, \quad \text{SWAP,} \quad \longrightarrow \quad \mathbf{b}_1 = \begin{bmatrix} F_{n-8} \\ \sim \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -F_{n-6} \\ \sim \end{bmatrix}, \\
 \mathbf{b}'_2 &= \mathbf{b}_2 + 3\mathbf{b}_1 = \begin{bmatrix} F_{n-10} \\ \sim \end{bmatrix}, \quad \text{SWAP,} \quad \longrightarrow \quad \mathbf{b}_1 = \begin{bmatrix} F_{n-10} \\ \sim \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} F_{n-8} \\ \sim \end{bmatrix},
 \end{aligned}$$

and so on. By choosing  $n$  to be arbitrarily large, we have constructed an example that requires arbitrarily many iterations to finish.

In conclusion, the number of iterations needed is fixed but arbitrarily large. Given any initial basis, it takes a fixed number of iterations to finish. However, given any number  $n$ , there exist bases that require more than  $n$  steps to reduce. In most cases, it takes very few iterations to finish; needing more is increasingly unlikely.

In the next two sections we examine the effects of using lattice reduction with traditional detectors. Let us use LR-ICD and LR-BLAST to refer to the detection schemes that combine lattice reduction with ICD and BLAST detection respectively.

## 3.4 Gaussian Channels

In this section we develop results for a fixed channel matrix  $\mathbf{H}$ .

### 3.4.1 Complexity

The incremental complexity inherent in the use of lattice reduction is determined by the number of iterations required to reduce the basis. As we saw in section 3.3.3, for  $2 \times 2$  channels, the number of iterations needed is small, less than two, for most channels. However, it is possible to construct examples that take arbitrarily many iterations to finish. The worst case is unbounded, but highly unlikely. Therefore, practically speaking, if we were to perform low complexity detection in this new basis as we proposed, the overhead associated with looking for the optimal basis would be very low. Thus, the overall algorithm has low complexity.

### 3.4.2 Performance

These new detection methods lead to decision regions (and thus performance) much closer to that of MLD, as we now develop.

Figure 3-4 shows a comparison of the decision regions for MLD and LR-ICD. It is drawn for a  $2 \times 2$  real example for illustration purpose. The MLD decision region is

a hexagon, and that of LR-ICD is a parallelogram. These regions also coincide with what are referred to as the *Voronoi cell* and *unit cell* of the lattice, respectively.

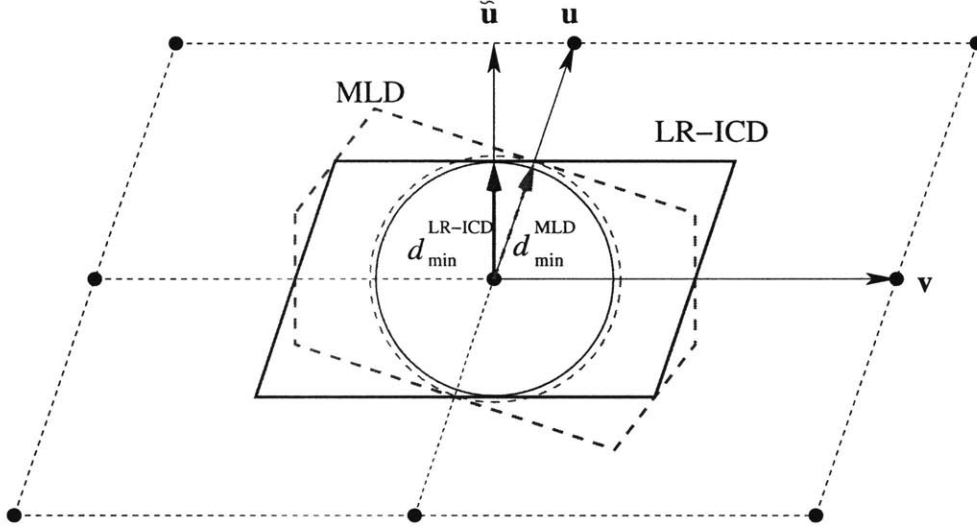


Figure 3-4: Comparison of the decision regions for MLD and LR-ICD. Minimum distances to the decision boundaries are also compared.

The minimum distances  $d_{\min}$  from a received constellation point to its decision boundaries are drawn. The length of  $d_{\min}$  is the minimum amount of noise needed for an error to occur, and determines the error probability at high SNR in white Gaussian noise, which is,  $2\mathcal{Q}(d_{\min}/\sigma_w)$ , where  $\sigma_w^2$  is the noise variance per dimension and  $\mathcal{Q}(x) = \int_x^\infty (1/\sqrt{2\pi}) \exp\{-x^2/2\}$ . We see that for LR-ICD,  $d_{\min}$  is shorter, so the performance is slightly worse. This is a result of the basis vectors not being exactly orthogonal. We now develop a precise bound on the ratio of  $d_{\min}^{\text{MLD}}$  to  $d_{\min}^{\text{LR-ICD}}$  to quantify the worst SNR gap to the MLD bound.

Generalizing Figure 3-4 to the complex case, we see that

$$d_{\min}^{\text{MLD}} = \frac{1}{2}\|\mathbf{u}\| \text{ and } d_{\min}^{\text{LR-ICD}} = \frac{1}{2}\|\tilde{\mathbf{u}}\|. \quad (3.10)$$

where

$$\begin{aligned}
\|\tilde{\mathbf{u}}\|^2 &= \|\mathbf{u}\|^2 - \left\| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\|^2 \\
&= \|\mathbf{u}\|^2 - \frac{\text{Re}\{\langle \mathbf{u}, \mathbf{v} \rangle\}^2}{\|\mathbf{v}\|^2} - \frac{\text{Im}\{\langle \mathbf{u}, \mathbf{v} \rangle\}^2}{\|\mathbf{v}\|^2} \\
&\geq \|\mathbf{u}\|^2 \left( 1 - \frac{1}{4} - \frac{1}{4} \right) = \frac{1}{2} \|\mathbf{u}\|^2.
\end{aligned} \tag{3.11}$$

Therefore,

$$d_{\min}^{\text{LR-ICD}} \geq \frac{1}{\sqrt{2}} d_{\min}^{\text{MLD}}, \tag{3.12}$$

which corresponds to a maximum SNR loss of 3 dB. This bound is tight; the worst case is achieved by, for example,  $\mathbf{u} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ , and  $\mathbf{v} = (\frac{1}{2} + \frac{1}{2}j) \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ . However, for many channel matrices the ratio is much closer to one.

For LR-BLAST,  $d_{\min}^{\text{LR-BLAST}} = \frac{1}{2} \min(\|\mathbf{u}\|, \|\tilde{\mathbf{v}}\|) \geq \frac{1}{2} \|\tilde{\mathbf{u}}\|$ , so it is never worse than LR-ICD. Comparing to MLD,  $d_{\min}^{\text{LR-BLAST}} = d_{\min}^{\text{MLD}}$ , when  $\|\tilde{\mathbf{v}}\| \geq \|\mathbf{u}\|$ , which happens quite often in the  $2 \times 2$  case. However, the worst-case ratio is still the same as the LR-ICD case.

In summary, LR can improve the performance of detection to within 3 dB from optimal in terms of  $d_{\min}$ . The actual gap depends on how well the particular channel can be reduced.

Another property of lattice reduction is that it monotonically improves detection performance. For both LR-ICD and LR-BLAST, each iteration of the reduction algorithm improves the decision region and increases  $d_{\min}$ . The more correlated the original basis vectors are, the greater the ultimate improvement. This behavior is illustrated by the following example channel matrices

$$\mathbf{H}_1 = \begin{bmatrix} 6 & 7 \\ 8 & -9 \end{bmatrix} \text{ and } \mathbf{H}_2 = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix},$$

whose resulting SNR gaps are listed in Table 3.1. Comparing the first two columns

to the last two, we see that little improvement is obtained for  $\mathbf{H}_1$ , which has nearly orthogonal columns, while a large improvement in dB is obtained for  $\mathbf{H}_2$ , which has highly correlated columns.

Table 3.1: SNR gaps to MLD performance for various detectors

	ICD	BLAST	LR-ICD	LR-BLAST
$\mathbf{H}_1$	0.31 dB	0.00 dB	0.31 dB	0.00 dB
$\mathbf{H}_2$	18.1 dB	17.0 dB	0.00 dB	0.00 dB

## 3.5 Rayleigh Fading Channels

In this section we develop results for ensembles of channels, i.e., for a random channel matrix  $\mathbf{H}$ . We focus on the Rayleigh fading case in which the entries of  $\mathbf{H}$  are independent and identically distributed  $\mathcal{CN}(0, 1)$  random variables, independent of the Gaussian noise.

### 3.5.1 Complexity

Since the incremental complexity is dependent on the realized channel, we plot in Figure 3-5 on both linear and logarithmic scales the empirical distribution of the number of iterations needed in the Rayleigh fading environment. Note that over 99% of the bases are reduced in two iterations or less, and that it becomes increasingly unlikely to need more iterations.

### 3.5.2 Performance

In Rayleigh fading, the average error probability  $P_e$  decays according to  $P_e \sim 1/\text{SNR}^\nu$  at high SNR, where  $\nu$  is the diversity order and reflects the system's tolerance of and robustness toward channel fading.

In the  $2 \times 2$  case, lattice reduction improves the diversity  $\nu$  achieved by ICD and BLAST detection to that of MLD. To see this, the average symbol error rate (SER)

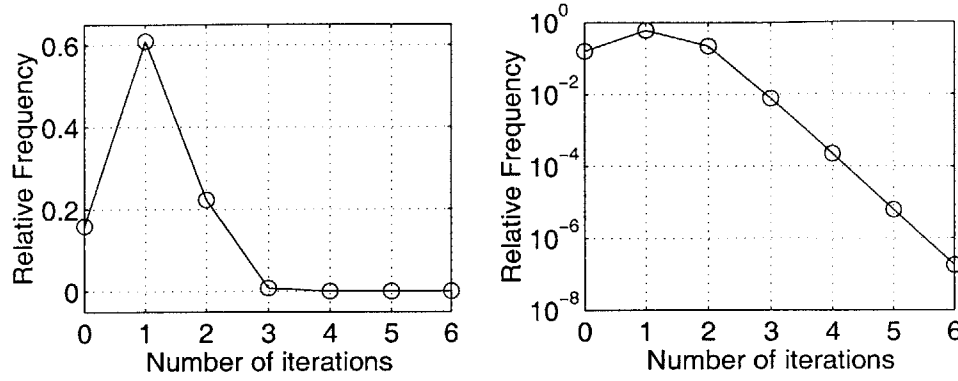


Figure 3-5: Distribution of number of iterations needed for  $2 \times 2$  lattice reduction.

curves for the various detection methods are plotted in Figure 3-6 for 16-QAM. The top two curves are for ICD and BLAST detection. In the high SNR regime, they both have diversity 1. Note that for BLAST, if there were no error propagation, the diversity for the entry detected second would have been 2. However, its actual diversity is only 1 due to error propagation from the entry detected first, which itself experiences only diversity 1.

The lowest curve is for MLD. The two curves immediately above it and parallel to it correspond to LR-ICD and LR-BLAST. In the high SNR regime, all three evidently have diversity two. This shows the improvement in diversity provided by using lattice reduction. Notice that with lattice reduction, the relative benefits of BLAST detection over ICD is smaller; this is a result of the basis vectors becoming more orthogonal.

It is also insightful to examine the empirical distribution of  $d_{\min}^2$  for these detectors, which is depicted in Figure 3-7. Relative to the original ICD and BLAST detection (dashed curves), it is evident that with lattice reduction (solid curves), the probability of having small  $d_{\min}^2$  is substantially reduced. Furthermore, comparing the LR curves to the MLD curve (dotted), we see that the performance gap is much less than the worst case 3 dB SNR loss. This is because channels yielding these larger losses are rare.

Figure 3-7 reflects the diversity behavior seen in Figure 3-6 from a different angle. The SER is related to a kind of “outage” probability, the probability of  $d_{\min}^2$  being



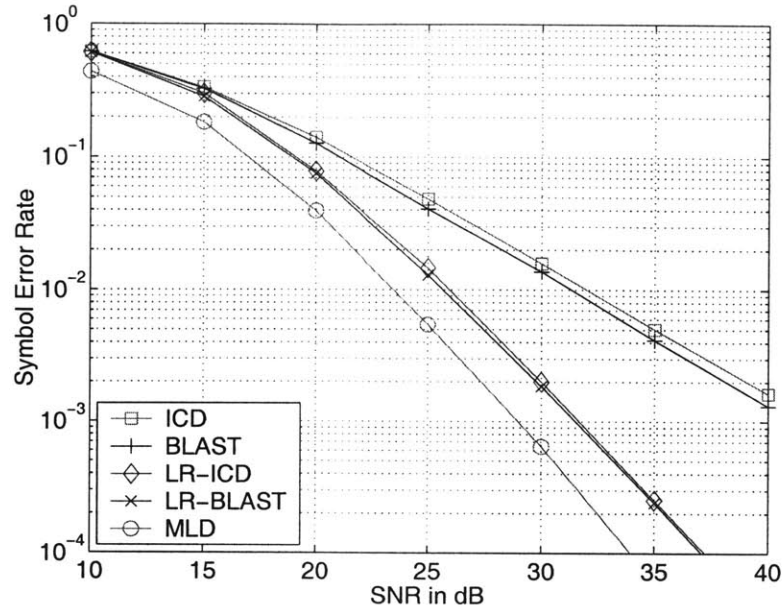


Figure 3-6: Symbol error rate curves for various detection methods in the  $2 \times 2$  complex case. The constellation used is 16-QAM.

less than a threshold, which is inversely related to SNR.

One feature in Figure 3-6 that is not captured by Figure 3-7 is the gap between the LR curves and the MLD curve. This is because the detection performance is also affected by the number of nearest neighbors and, indirectly, the size of the constellation. In a finite constellation, some points have fewer nearest neighbors, for example, the edge points. In some extreme cases, it is even possible for a point to have all its nearest neighbors distance  $d_{\min}$  away to be outside the constellation, in which case, the effective  $d_{\min}$  is actually greater. For these reasons, LR based detection, which treats the constellation as an infinite lattice, is slightly further sub-optimal compared to MLD, which takes advantage of the finite size of the constellation. However, as the constellation gets larger, these difference diminish. This finite constellation effect can be seen by extending the constellation to 64-QAM and 256-QAM from the original 16-QAM, at the 25dB noise level. The corresponding SER curves for MLD are plotted in Figure 3-8 together with the corresponding SER of LR-BLAST. We can see that as the constellation gets larger, the gap between MLD and LR-BLAST

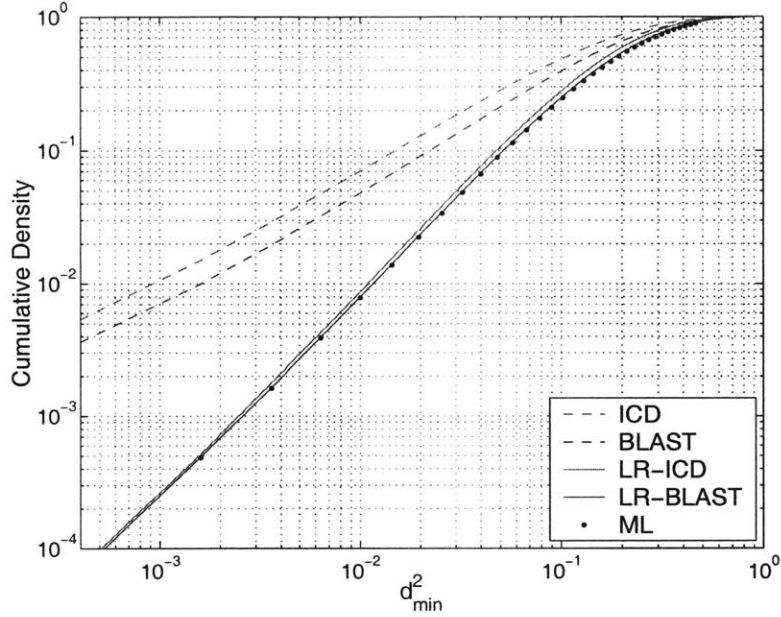


Figure 3-7: Comparisons of the cumulative density of  $d_{\min}^2$ .

becomes smaller.

### 3.5.3 Diversity-Multiplexing Tradeoff

In this section, we numerically evaluate the diversity-multiplexing tradeoff achieved using the proposed lattice-reduction-aided detectors in an uncoded system. We show that the best tradeoff achievable by any length-one code is effectively achieved.

To numerically evaluate the diversity-multiplexing tradeoff, we perform simulations with the LR-BLAST detector for rates  $R = 4, 8, 12, \dots, 32$  b/s/Hz using constellations with sizes per dimension  $M = 2, 4, 8, \dots, 256$ . The resulting family of  $2 \times 2$  block error rate curves for the various rates are plotted in Figure 3-9. The outage probability curves for those rates shown earlier in Figure 2-5 of section 2.3.4 are re-plotted here as light gray lines for comparison.

The diversity and multiplexing gains achieved can be numerically measured from the slopes of the error rate curves and the horizontal spacings between these curves, as discussed earlier in section 2.3. We see that the limiting slope of each curve

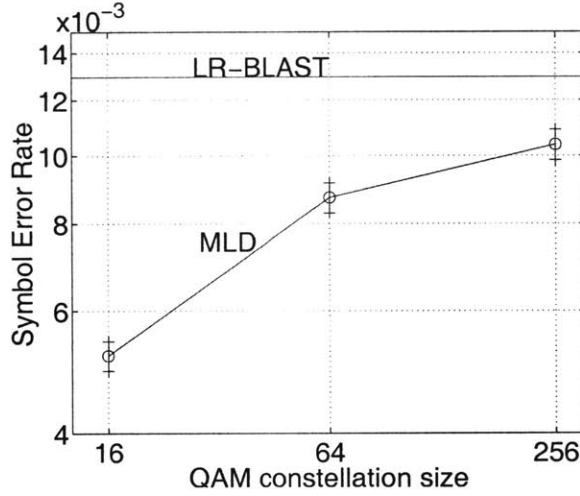


Figure 3-8: As constellation size grows, the gap between the symbol error rates of MLD and LR-BLAST diminishes. The noise level is such that the SNR is 25 dB for the 16-QAM constellation.

is 2. This is the maximum diversity gain achieved. The horizontal gaps between the curves with rate differential 4 b/s/Hz is approximately 6 dB. This implies a maximum multiplexing gain of 2 b/s/Hz per 3 dB. The family of curves appears parallel, so we can get any linear combination of the maximum diversity and multiplexing gains. Therefore, the diversity-multiplexing tradeoff achieved is a straight line between  $(0, 2)$  and  $(2, 0)$ , i.e.,  $d(r) = 2 - r$ .

Zheng and Tse showed in [41] that the best diversity-multiplexing tradeoff achievable by any length-one code, which they refer to as *space-only* code, is a straight line between  $(0, N_r)$  and  $(\min(N_t, N_r), 0)$ . Substituting in  $N_t = N_r = 2$ , we have  $d(r) = 2 - r$ . Therefore, a system with uncoded transmission and lattice-reduction-aided detector can effectively achieve the best tradeoff achievable by any space-only code.

One implication of the above result is that an uncoded system (with near optimal decoding) is just as good as any other space-only coded system. The intuition for this is that a linear space-only code can be represented as  $\mathbf{x} = \mathbf{G}\mathbf{s}$ , where  $\mathbf{G}$  is some code generating matrix and  $\mathbf{s}$  is a vector of uncoded symbols. By writing  $\mathbf{H}\mathbf{x} = \mathbf{H}\mathbf{G}\mathbf{s} = (\mathbf{H}\mathbf{G})\mathbf{s}$ , the code matrix  $\mathbf{G}$  can be absorbed by the channel matrix  $\mathbf{H}$ .

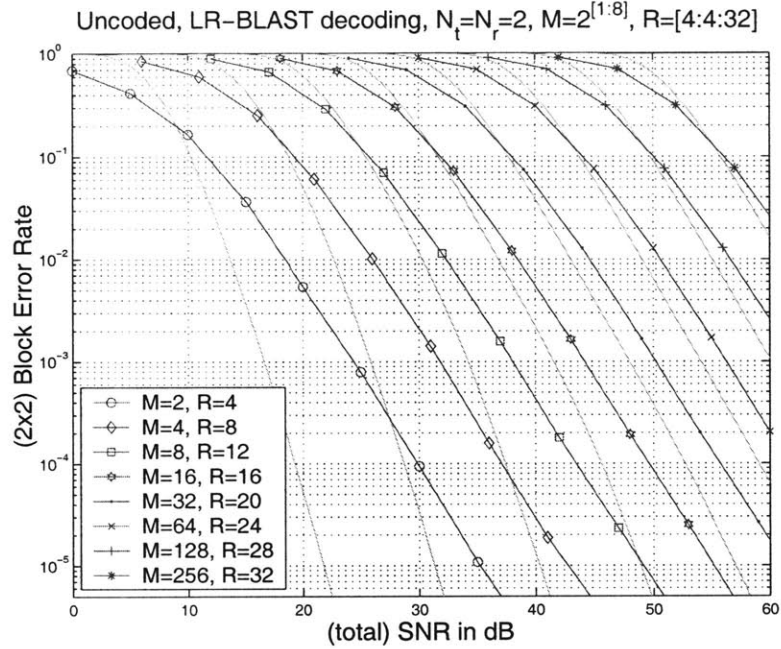


Figure 3-9: Uncoded system with LR-BLAST decoder. The maximum slope reached is 2. The horizontal spacings between the curves are 6 dB.

Therefore,  $\mathbf{G}$  has little effect except possibly changing the statistics of the effective channel in some way.

We note that if longer codes are allowed, then better diversity-multiplexing trade-offs can be achieved. We see that the set of unmarked light gray curves in Figure. 3-9, which represent the ultimate performance achievable by infinitely long codes, have better slopes.

### 3.6 Lattice Reduction at Transmitter

Another use of lattice reduction techniques in addition to the detection problem we just discussed is to apply them at the transmitter for power reduction when transmitter has knowledge of the channel and pre-compensate for it. In this section, we briefly discuss this problem from a geometric perspective. We first describe a naive way of pre-compensating for the channel, then present an idea of using a lattice to represent messages, which leads to the application of lattice reduction techniques. We

only illustrate the basic ideas, the details are left for future development.

When the transmitter has knowledge of the channel, it can pre-compensate for the distortion by transmitting  $\mathbf{H}^{-1}\mathbf{x}$  instead of  $\mathbf{x}$ . The resulting received signal is then  $\mathbf{y} = \mathbf{H}(\mathbf{H}^{-1}\mathbf{x}) + \mathbf{w} = \mathbf{x} + \mathbf{w}$ . This means that the receiver effectively sees an AWGN channel, and can detect each entry of  $\mathbf{x}$  independently without knowing the channel. This idea is illustrated in Figure 3-10.

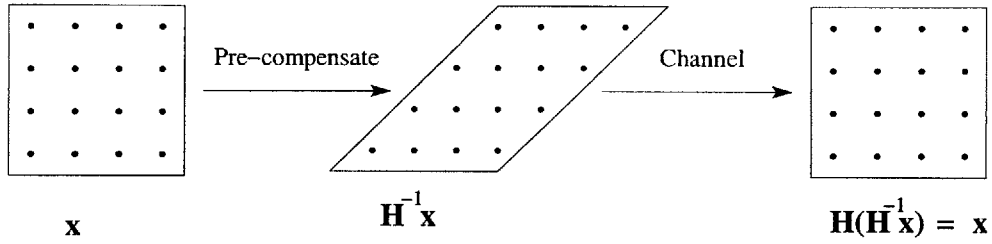


Figure 3-10: When the transmitter knows the channel, it can pre-compensate for the distortion by transmitting  $\mathbf{H}^{-1}\mathbf{x}$ , so that the received constellation is the original one.

One problem with pre-multiplying with  $\mathbf{H}^{-1}$  is that the resulting constellation region becomes very elongated, as seen in Figure 3-10. (This effect is similar to the noise enhancement problem of the inverse channel detector in section 3.2.) Having an elongated constellation region is inefficient in terms of power usage because it takes more power to transmit points further from the origin. This suggests that the constellation region need to be made more circular.

Next, we review an idea of using a set of congruent points, or a lattice, to signal a message. Later, this will allow us to use lattice reduction techniques to make the constellation region more circular and reduce the transmit power.

This idea was introduced by Tomlinson and Harishima as part of their transmitter pre-coding algorithm [19]. A set of points that are congruent modulo the constellation region are used to represent the same message. as illustrated in Figure 3-11. All points marked by “o” are congruent to the point marked by “+” modulo the constellation region drawn with solid lines. To transmit the message originally represented only by “+”, we can now use any of the “o”s. Note that the set of “o”s together with the “+” form a lattice. Among all the lattice points, we would pick the one closest to

the origin to minimize transmit power. At the receive, if we receive any “o”, which is outside the constellation region, we find its congruent image inside the constellation region, which is the “+”, and treat it as if the “+” is actually received.

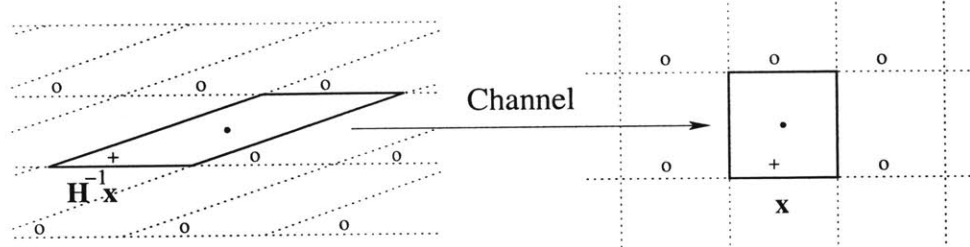


Figure 3-11: At the transmitter, all points that are congruent modulo the constellation region, which form a lattice, are used to represent the same message. Points labeled “o” represent the same message as the point labeled “+”. At the receiver, any “o” can be mapped back to “+” via modulo operations.

With this idea of using a lattice to represent the same message, the problem of minimizing transmit power becomes the problem of finding the nearest point of a shifted lattice to the origin. This allows us to use lattice reduction techniques.

In Tomlinson-Harishima pre-coding, modulo operations at the transmitter are performed one dimension at a time, similar to the successive cancellation technique used in BLAST detection. We will not review it here. It suffices to say that the resulting constellation region has the same shape as the decision region achieved by BLAST detection for the same channels shown in Figure 3-1. It improves the power efficiency but is not optimal.

Let us now look at what constellation region is the most desirable. We can consider the original transmitted constellation region and its periodically extended copies, which are drawn in Figure 3-11 with dotted lines, as unit cells of a lattice. The set of “o”s is a shifted version of this lattice. We see that there is exactly one “o” in each unit cell. In fact, if we were to consider other unit cells of this lattice, there would still be exactly one “o” in each cell, no matter what the unit cell is. Therefore, by choosing different unit cells as the constellation region, we choose which “o” to transmit. The best constellation region to use is the unit cell with the least second

moment, or energy, which is the Voronoi cell of the lattice.

Finding the Voronoi cell is difficult, instead, lattice reduction techniques can be applied to obtain constellation regions close to the Voronoi cell. In particular, a set of basis vectors that are shorter and more orthogonal would lead to a unit cell that is more square.

An example of using a constellation region with lattice-reduced basis is illustrated in Figure 3-12. If we want to transmit a point in the region labeled 1 in the original constellation region associated with transmitting  $\mathbf{H}^{-1}\mathbf{x}$ , we should instead transmit its congruent image in the region labeled 1' to reduce power. At the receiver, if a point in region 1' is received, it is mapped back to region 1 using modulo operations. Similar procedures take place for regions labeled 2, 3, and 4. Comparing the elongated parallelogram and the square, we see that transmission power is much reduced.

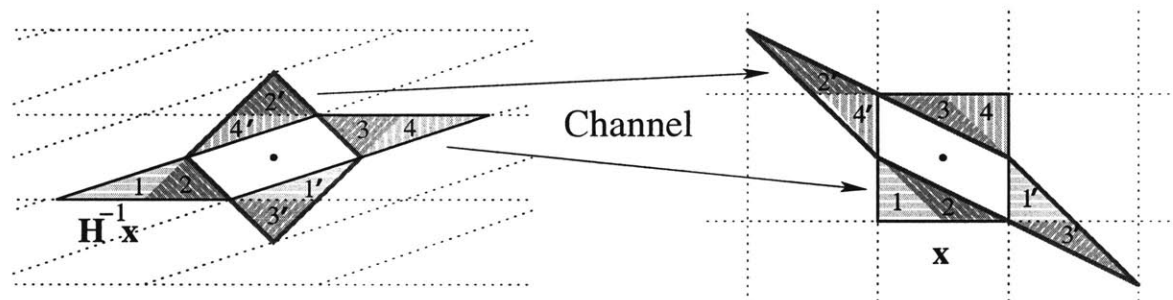


Figure 3-12: Treat the original transmitted constellation region,  $\mathbf{H}^{-1}\mathbf{x}$ , as a unit cell of a lattice. Power reduction can be achieved by using a more square unit cell corresponding to a different basis as the transmitted constellation region. Regions shaded in the same way and labeled using the same number are congruent to each other and represent the same set of messages.

This illustrates the basic idea behind using lattice reduction techniques for transmitter pre-coding power reduction. The details of this algorithm are left for future development.

## 3.7 Higher Dimensional Lattice Reduction

In the previous sections, we have demonstrated how using lattice reduction can improve the performance of traditional low complexity detectors to be close to that of the maximum likelihood detector. We also illustrated how LR can be used at transmitter for pre-coding as well. LR may also potentially be applied to other problems such as source coding, when the quantization points form a lattice.

The lattice reduction techniques we presented are for the case of two (complex) dimensions only. We now investigate how feasible and how useful LR might be in higher dimensional cases. We address issues including what basis should be considered optimal, what lattice reduction algorithms can be used, what their complexity levels are, and how well they work when combined with traditional detectors. The main goal is to point out existing work on lattice reduction theory and discuss them.

Generally speaking, the lattice reduction problem is NP-hard in the dimension of the lattice. Conway and Sloane [6] expressed their feeling toward this type of problems as, anything associated with high dimensional lattice is hard except finding its determinant (the volume of a unit cell). For example, finding the covering radius has proven to be NP-hard, while finding the packing radius or the shortest vector is conjectured to be NP-hard.

In this section, we discuss several lattice reduction algorithms including the sub-optimal polynomial time LLL algorithm, for which, we also use numerical simulation to demonstrate the complexity and performance when combined with ICD and BLAST detectors.

### 3.7.1 Existing Algorithms

We discuss three different notions of lattice reduction, Minkowski reduced form, Korkin-Zolotarev (K-Z) reduction, and Lenstra, Lenstra, and Lovász (LLL) reduction algorithm. There are many other reduction algorithms. An extensive list of references can be found on page 41 of the well-known textbook by Conway and Sloane [6].



## Minkowski Reduced Form

In the two dimensional case we studied, we identified the optimal basis to be  $(\mathbf{u}, \mathbf{v})$ , where  $\mathbf{u}$  is the shortest (non-zero) vector in the lattice and  $\mathbf{v}$  is the shortest vector that is not a multiple of  $\mathbf{u}$ . This description of optimal basis can be extended to higher dimensions and is known as the Minkowski reduced form [6].

More formally, a basis  $\mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$  is *Minkowski reduced* if each basis vector  $\mathbf{b}_i$  is the shortest vector that is not a linear combination of  $\mathbf{b}_1, \dots, \mathbf{b}_{i-1}$ .

There are no polynomial time algorithms for finding the Minkowski reduced basis since even finding the shortest vector is believed to be NP-hard. What is known about the Minkowski reduced form is a set of conditions to check whether a given basis is reduced, similar to the conditions we obtained for the two-dimensional case in Lemma 3.3. These conditions exist for dimensions up to 8, and can be found in [6] and the references there-in. The conditions are expressed as sets of inequalities between the lengths and the correlations of the basis vectors. As the number of dimensions grows, the number of conditions increases with it, and the functional form of the conditions also becomes more complex.

## Korkin-Zolotarev Reduction

The Korkin-Zolotarev, or K-Z, reduction form [15] is similar to the Minkowski reduced form in the sense that the basis is defined to be a series of short basis vectors. The first basis vector  $\mathbf{b}_1$  is also the shortest vector of the lattice. The difference is that after the basis vectors  $\mathbf{b}_1, \dots, \mathbf{b}_{i-1}$  are chosen, the next one,  $\mathbf{b}_i$  is chosen not to minimize its length, but to minimize the length of its component orthogonal to  $\mathbf{b}_1, \dots, \mathbf{b}_{i-1}$ . That is,  $\|\tilde{\mathbf{b}}_i\|$  is minimized instead of  $\|\mathbf{b}_i\|$ .

Just as for Minkowski reduction, there is no polynomial time algorithm for K-Z reduction. The fastest known algorithm for K-Z reduction algorithm for a basis with integer entries is due to Schnorr [32].

In [2] and some references there-in, K-Z reduction is used for the purpose of lattice decoding, i.e., finding the nearest lattice point to a given point. However, in their

study, the lattice is fixed. They did not focus on the complexity of finding the K-Z reduced basis, but only on how to use the already reduced basis for lattice decoding.

## LLL Reduction

The LLL lattice reduction algorithm by Lenstra, Lenstra, and Lovász [22] is a polynomial time algorithm that provides a set of basis vectors that are generally short. In particular, the shortest vector it finds is shorter than a certain multiple of the true shortest vector. The algorithm was originally developed for integer programming [21] and factoring polynomials with rational coefficients [20].

A detailed description of the LLL algorithm can be found in [22]. We now briefly summarize the procedure and the bounds on the lengths of the resulting basis vectors.

The LLL algorithm is a more general version of the iterative reduction algorithm we proposed in section 3.3.2. It also iterates between two steps, subtracting integer copies of some vectors out of others to reduce correlation and swapping vectors so that the shorter ones tend to have smaller indexes. These two steps take place iteratively until no changes are made.

More specifically, given a basis,  $\mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ , let  $\mathbf{b}_i(j)$  be the component of  $\mathbf{b}_i$  orthogonal to  $\mathbf{b}_1, \dots, \mathbf{b}_{j-1}$ . Using Gram-Schmidt orthogonalization, we can write

$$\mathbf{b}_i = \sum_{j=1}^i \mu_{ij} \mathbf{b}_j(i), \quad i = 1, \dots, n, \quad (3.13)$$

The coefficients  $\mu_{ij}$ ,  $1 \leq j < i \leq n$ , are related to the correlation between the basis vectors. We also have  $\prod_{i=1}^n \|\mathbf{b}_i(i)\| = \det(\mathbf{B})$ .

The goal of the first step of each iteration is to make all  $|\mu_{ij}| \leq \frac{1}{2}$ . Bases with this property are called *weakly reduced*. We can reduce each  $|\mu_{ij}| \geq \frac{1}{2}$  by subtracting  $\lfloor \mu_{ij} \rfloor$  (nearest integer to  $\mu_{ij}$ ) copies of  $\mathbf{b}_j$  out of  $\mathbf{b}_i$ . To maximize efficiency, we should perform this subtraction from  $i = 2$  to  $n$  and from  $j = i - 1$  to 1.

Once the basis is weakly reduced, the second step of each iteration involves looking

for the index  $i$  violating

$$\|\mathbf{b}_i(i)\|^2 \leq \frac{4}{3}\|b_{i+1}(i)\|^2, \text{ for } 1 \leq i < n \quad (3.14)$$

and swapping  $\mathbf{b}_i$  and  $\mathbf{b}_{i+1}$ . This helps bringing the shorter vectors forward, so that they can be used to further reduce other vectors during the next iteration. The coefficient  $4/3$  is there to ensure faster convergence. It can be replaced by any number larger than 1 but less than  $3/2$ .

It can be shown that the algorithm that iterates between the two steps described above is polynomial time and that the resulting reduced basis has the following properties, (assuming the factor of  $4/3$  is used,)

1.  $\|\mathbf{b}_1\| \leq 2^{(n-1)/2}\xi$ , where  $\xi$  is the length of the shortest vector of the lattice;
2.  $\|\mathbf{b}_1\| \leq 2^{(n-1)/4} \sqrt[n]{\det(\mathbf{B})}$ ;
3.  $\|\mathbf{b}_1\| \cdots \|\mathbf{b}_n\| \leq 2^{n(n-1)/4} \det(\mathbf{B})$ .

### 3.7.2 Complexity and Performance of LLL

In this section, we use numerical simulations to study the complexity of the LLL algorithm and see how well it would work when combined with the traditional ICD and BLAST detectors. We show that the complexity measured by the number of iterations needed increases rather rapidly with the number of dimensions, and the performance gap to the ML detector increases as well.

The original LLL algorithm is developed for real matrices instead of complex. We can always choose to treat one complex dimension as two real dimensions, unless we want to take advantage of the special orthogonality relationship between each pair of real and imaginary components. In this section, we simply work with the real case to obtain some intuitions.

Also, when the number of dimensions is large, it is difficult to discuss the number of iterations needed for specific matrices, like in Figure 3-3. Instead, we look at how many iterations are needed for most channels, more specifically, the distribution of

the number of iterations needed for the case where the basis matrix has IID Gaussian entries.

For dimensions,  $n = 2, 4, 6$ , and  $8$ , we perform LLL lattice reduction as described earlier for  $n \times n$  real matrices,  $\mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ , randomly generated with IID zero-mean, unit variance, Gaussian distributions. We record the number of iterations taken and the lengths of the resulting basis vectors. The empirical probability distribution, as well as the cumulative distribution, of the number of iterations taken are plotted on log-scales in Figure 3-13 and Figure 3-14, respectively. Note that these are the actual numbers of iterations taken, not an upper bound, as complexity theory provides.

We see from the figures that for higher dimensions, the number of iterations needed increases rapidly. To get a better sense of the increase in complexity, let us look at some specific numbers. In the  $2 \times 2$  (real) case, over 99% of the time, it takes two iterations or less to reduce the basis. For  $n = 4, 6$ , and  $8$  dimensions, the 99 percentile point becomes 11, 25, and 43 iterations, which can be read from Figure 3-14. The probability of needing more iterations decreases exponentially with the number of iterations, but slower for higher dimensions. The average number of iterations taken is 0.7, 4.4, 10.7, and 19.1 respectively.

Another thing to note is that the amount of computation associated with each iteration also increases with the number of dimensions. In particular, during the first step of making the basis weakly reduced, there are up to order  $n^2$  many  $\mu_{ij}$ 's to reduce. Reducing each one also requires scaling and addition of length  $n$  vectors.

Next, let us look at how well the LLL reduced basis work with the traditional ICD and BLAST detectors, compared to the maximum likelihood detector. The performance measure we use is the magnitude of the minimum amount of noise necessary for an error to occur, i.e., the radius of the largest sphere inside the decision region of each detector, similar to the ones drawn in Figure 3-4. Let us denote these radii with  $d_{\min}^{\text{LR-ICD}}$ ,  $d_{\min}^{\text{LR-BLAST}}$ , and  $d_{\min}^{\text{ML}}$ , for the three detection methods, respectively.

If we use the BLAST detector, i.e., employ the successive cancellation method in (3.2), then the decision region is rectangular, as shown in Figure 3-1 (c). The lengths

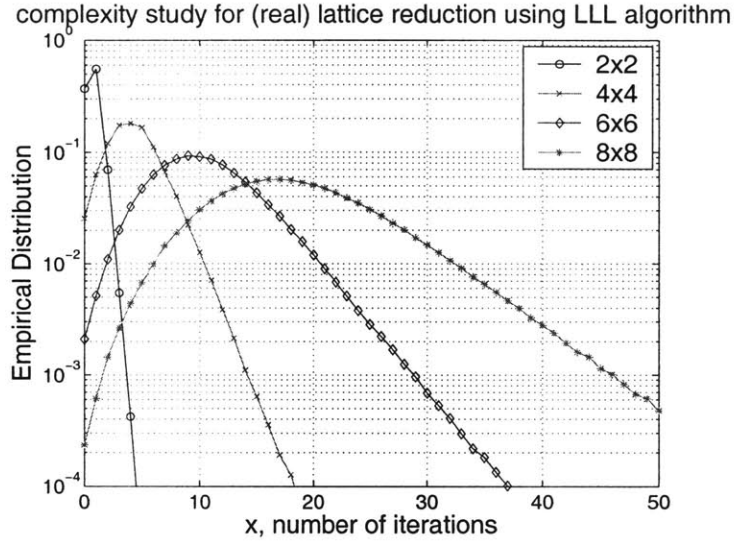


Figure 3-13: Empirical distribution of number of iterations needed for  $n \times n$  (real) lattice reduction using the LLL algorithm for the cases of  $n = 2, 4, 6, 8$  dimensions.

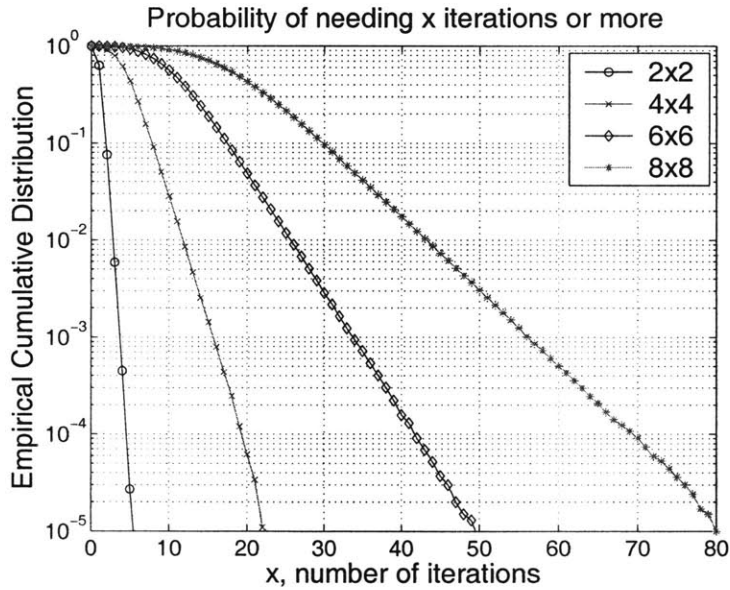


Figure 3-14: Empirical cumulative distribution, indicating probability of needing  $x$  iterations or more.

of the sides are  $\|\mathbf{b}_i(i)\|$ , the component of  $\mathbf{b}_i$  orthogonal to  $\mathbf{b}_1, \dots, \mathbf{b}_{i-1}$ . Instead, we have  $d_{\min}^{\text{LR-BLAST}} = \frac{1}{2} \min_i \|\mathbf{b}_i(i)\|$ . For the ICD detector, we need to look at  $\|\tilde{\mathbf{b}}_i\|$ , the component of  $\mathbf{b}_i$  orthogonal to all other vectors, not just the previous ones. Therefore,  $d_{\min}^{\text{LR-ICD}} = \frac{1}{2} \min_i \|\tilde{\mathbf{b}}_i\|$ . In the case of ML detector, the corresponding measure is the length of the true shortest vector of the lattice,  $\xi$ . We have  $d_{\min}^{\text{ML}} = \frac{1}{2}\xi$ . When the number of dimensions is not too large, the shortest vector of a lattice can be found using either brute force search or a more efficient technique called sphere decoding [26, 38].

For dimensions  $n = 2, 4, 6$ , and  $8$ , we plot the empirical distribution of the ratio  $d_{\min}^{\text{LR-BLAST}}/d_{\min}^{\text{ML}}$  and  $d_{\min}^{\text{LR-ICD}}/d_{\min}^{\text{ML}}$  for LLL reduced bases in figure Figure 3-15 and Figure 3-16, respectively.

We see that as the number of dimensions increases, the distribution of the ratio moves down from 1, meaning that LR-BLAST and LR-ICD become further away from optimal. The worst case ratio also moves down. After  $10^5$  trials, the empirical worst case found for LR-BLAST is 1.7, 3.3, 4.0, and 6.1 dB from optimal for  $n = 2, 4, 6, 8$  dimensions. For LR-ICD, the gaps are 1.7, 4.3, 6.4, and 8.8 dB. Compared to LR-ICD, LR-BLAST not only performs better on average, but also seems to have better worst case bound.

One technical detail to note is the  $4/3$  factor in (3.14). If we were to use other values between  $4/3$  and 1, then we would get better reduced basis at the expense of increased complexity.

## 3.8 Summary

In this chapter, we studied uncoded MIMO communication systems and proposed new coherent detection methods. By incorporating lattice reduction, these methods significantly improve the performance of traditionally employed low-complexity detectors, in particular, ICD and BLAST detectors. We investigated the case of the two-transmit two-receive antenna systems in detail. We presented an iterative lattice reduction algorithm for finding the optimal basis and studied its complexity. We

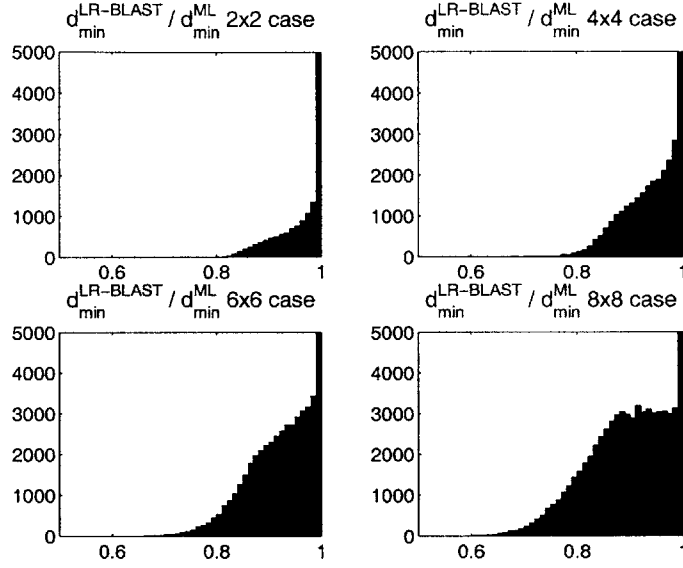


Figure 3-15: Performance of LR-BLAST detectors using the LLL algorithm, compared to that of the ML detector, for  $n = 2, 4, 6, 8$  dimensional cases. The ratio  $d_{\min}^{\text{LR-BLAST}} / d_{\min}^{\text{ML}}$  indicates how far LR-BLAST is away from optimal.

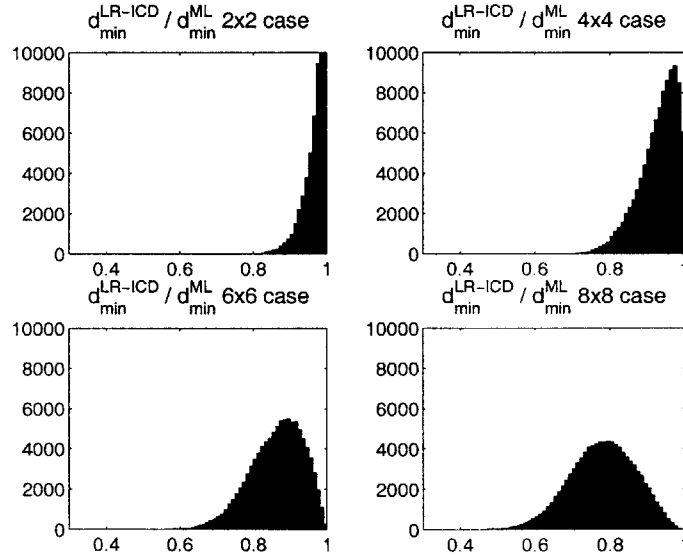


Figure 3-16: Performance of LR-ICD detectors using the LLL algorithm, compared to that of the ML detector, for  $n = 2, 4, 6, 8$  dimensional cases.

showed that the number of iterations needed is typically small and it is increasingly unlikely to need more. We also showed that, relative to optimal MLD, LR techniques is sub-optimal by no more than 3 dB in terms of SNR for any Gaussian channel, and allows us to achieve the same diversity on the Rayleigh fading channel, assuming sufficiently large constellations are used.

While the proofs and simulations in this study are mostly limited to the  $2 \times 2$  case, for higher dimensional cases, lattice reduction ideas can still be applied. However, the complexity increases, as well as the average and worst-case gap to MLD. So generally speaking, this lattice reduction idea is mainly meant for applying to low dimensional cases.

One shortcoming of lattice decoding is that the constellation is treated as an infinite lattice, so there is a boundary issue. When the received signal falls outside of the valid constellation region, the nearest lattice point found may not be a valid codeword. This would lead to errors that could be avoided by MLD.

Extending lattice reduction techniques to transmitter pre-coding can lead to additional benefits. In this work, we briefly illustrated the basic ideas graphically; the details are left for future development. If we were also allowed to transmit at different rates for each entry of  $\mathbf{x}$  and the objective were to maximize the total rate, we might also want to employing water-filling techniques.



# Chapter 4

## Structured Codes with Minimum Delay

### 4.1 Introduction

In this chapter, we investigate the problem of using short structured space-time block codes to achieve the optimal diversity-multiplexing tradeoff in the case of two-transmit two-receive antenna systems, and try to understand what is fundamentally possible. For this case, the optimal tradeoff was examined in section 2.3.3 and plotted in Figure 2-4.

The primary question of interest here is whether the optimal tradeoff can be achieved using length-two codes. From the rank criterion in Lemma 2.3 in section 2.4, we see that it is necessary to have  $T \geq N_t = 2$  to achieve full diversity. In section 2.5, we see that Gaussian random codes with code length  $T \geq 3$  can achieve the optimal tradeoff, while those with  $T = 2$  can not. In this chapter, we answer this previously open question by presenting a length-two code which we call tilted-QAM code that can in fact achieve the optimal tradeoff.

The system model we use in this chapter is  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , where  $\mathbf{X}$  is the  $2 \times 2$  transmitted signal matrix,  $\mathbf{H}$  is the  $2 \times 2$  channel matrix,  $\mathbf{W}$  is the additive white Gaussian noise and  $\mathbf{Y}$  is the received signal. Under the Rayleigh fading model, the entries of  $\mathbf{H}$  are independent and identically distributed  $\mathcal{CN}(0, 1)$  random variables,

and are assumed to be known by the receiver, but not the transmitter.

We first review a well-known length-two code called the *orthogonal space time block code* in section 4.2. OSTBC is a well structured code and is highly attractive for its low decoding complexity. It uses a smart repetition to ensure all its difference matrices are full rank, thus achieving the maximum diversity gain. However, this repetition causes a loss of multiplexing gain. The tradeoff it achieves is below that of the length-two Gaussian random code.

In the rest of this chapter, we develop the tilted-QAM coding scheme. In section 4.3, we introduce the design of the tilted-QAM code, which improves upon OSTBC by replacing the repetition with a suitably chosen rotation.<sup>1</sup> Using the criterion of maximizing the worst case determinant, we identify a set of rotation angles that is universally optimal and leads to the same worst case determinant for all rates. In section 4.4, we analyze the performance of the tilted-QAM code from two perspectives, and show that our design can indeed achieve the optimal diversity-multiplexing tradeoff. We believe that having the worst case determinant maintaining a non-vanishing distance away from zero as rate increases is important for obtaining the optimal tradeoff. In section 4.5, we numerically simulate the performance of the tilted-QAM code to demonstrate that the optimal tradeoff is effectively achieved. In section 4.6, we discuss applying the tilted-QAM code design idea to a single antenna fading problem.

## 4.2 OSTBC

An existing well-known space-time code is OSTBC, first introduced by Alamouti in [1] for the two transmit and any number of receive antennas case, and then extended by Tarokh in [34] for more general cases. In this section, we first describe the smart repetition structure of OSTBC and then evaluate the diversity-multiplexing tradeoff achieved. We also present numerical simulation results at the end.

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<sup>1</sup>Interestingly, such rotation ideas are also used by Boutros and Viterbo [3] in their design of codes for single antenna fading channels.

### 4.2.1 The Smart Repetition

OSTBC encodes two information symbols,  $s_1$  and  $s_2$ , into one  $2 \times 2$  transmitted signal matrix  $\mathbf{X}$  in the following fashion,

$$\mathbf{X} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}, \quad (4.1)$$

where,  $(\cdot)^*$  indicates conjugation. We see that it effectively transmits each of the two symbols twice, using two antennas in two time slots.

The resulting received signal is

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}. \quad (4.2)$$

We can rearrange terms and conjugate  $y_{12}$  and  $y_{22}$  to obtain the effective channel

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{12}^* \\ y_{22}^* \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{12}^* & -h_{11}^* \\ h_{22}^* & -h_{21}^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} w_{11} \\ w_{21} \\ w_{12}^* \\ w_{22}^* \end{bmatrix}. \quad (4.3)$$

The effective channel vectors,  $\begin{bmatrix} h_{11} & h_{21} & h_{12}^* & h_{22}^* \end{bmatrix}^T$  and  $\begin{bmatrix} h_{12} & h_{22} & -h_{11}^* & -h_{21}^* \end{bmatrix}^T$ , are orthogonal to each other. Therefore, there is no interference between  $s_1$  and  $s_2$ . Component-wise decoding can be easily done. Low complexity is one of the major advantages of OSTBC.

### 4.2.2 Theoretical Performance Analysis

In this section, we examine the diversity-multiplexing tradeoff achieved by OSTBC.

By using repetition to spread each symbol across space and time, OSTBC can achieve the maximum diversity. This can be shown using Lemma 2.3, according to which, we need to verify that all difference matrices are full rank, i.e., have non-zero

determinant.

Without loss of generality, let us fix one of the code matrices to be  $\mathbf{0}$ , so we can look at the non-zero codeword matrices instead. We have

$$\det(\mathbf{X}) = |s_1|^2 + |s_2|^2 \neq 0 \text{ when } \mathbf{X} \neq \mathbf{0}. \quad (4.4)$$

Therefore, OSTBC achieves the maximum diversity of  $N_t N_r = 4$  when  $r = 0$ , i.e., error probability decays like  $\text{SNR}^{-4}$  when rate is kept constant.

However, due to the repetition, only one new symbol is transmitted at a time, so it can only achieve  $r = 1$  when  $d = 0$ , i.e., for a fixed target error probability,  $R$  increases by one for every 3 dB increase in SNR. Zheng and Tse [41] showed that the diversity-multiplexing tradeoff achievable by an OSTBC system is a straight line between  $(r, d) = (0, 4)$  and  $(1, 0)$  as shown in Figure 4-1. The optimal tradeoff curve and that achieved by the length-two Gaussian random codes are also plotted.

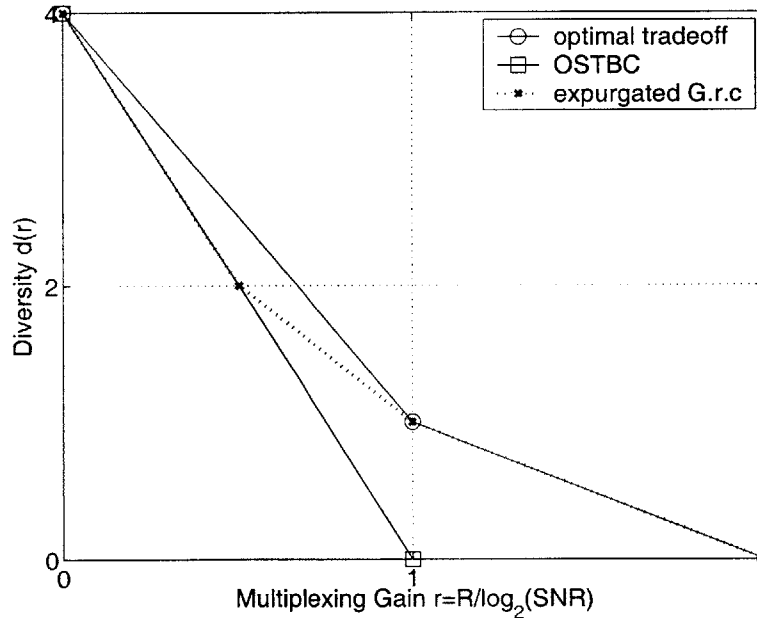


Figure 4-1: Diversity-multiplexing tradeoff achieved by orthogonal space-time block code, compared with the optimal tradeoff and that of the expurgated Gaussian random code, for the case  $N_t = N_r = T = 2$ .

We see that OSTBC does not achieve the optimal diversity-multiplexing tradeoff

curve. For  $r > 0.5$ , it is also inferior than the length-two Gaussian random code. Most importantly, the maximum multiplexing gain achievable by OSTBC is only  $r = 1$ . This implies that to transmit at a reasonably high rate, unnecessarily high SNR is needed.

### 4.2.3 Simulation Results

In this section, we demonstrate the performance of OSTBC using numerical simulations, which is set up as follows. Two uncoded information symbols  $s_1, s_2$  chosen out of QAM-like constellations are encoded into a  $2 \times 2$  transmitted signal matrix  $\mathbf{X}$  according to (4.1). The matrix  $\mathbf{X}$  is then transmitted over a  $2 \times 2$  multiple antenna channel,  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ . Random channels with IID  $\mathcal{CN}(0, 1)$  entries are generated for each trial. At the receiver, ML decoding is easily implemented, because the effective channels are orthogonal.

We perform simulations at rates  $R = 4, 8, 12, 16$  b/s/Hz using constellations with sizes per dimension,  $M = 4, 16, 64, 256$ . We note that  $R = 1 \cdot \log_2(M^2)$ , because only one new symbol is transmitted at a time due to the repetition. The resulting family of  $2 \times 2$  block error rate curves for the various rates are plotted in Figure 4-2. The outage probability curves for those rates are also plotted for comparison.

For OSTBC, we see that the slope of each curve approaches 4, which is the maximum diversity gain. The horizontal gaps between the curves with rate differential 4 b/s/Hz is approximately 12 dB. This implies a maximum multiplexing gain of 1 b/s/Hz per 3 dB. Compared to the underlying outage probability curves, OSTBC becomes further from optimal as rate increases. This is the result of the loss of multiplexing gain.

OSTBC is sub-optimal mainly because it is fundamentally a repetition code. Although the repetition is what allows OSTBC to achieve the maximum diversity gain, it reduced the maximum multiplexing gain to 1. Next, we propose an alternative scheme that overcomes this shortcoming by replacing the repetition with a suitable chosen rotation.

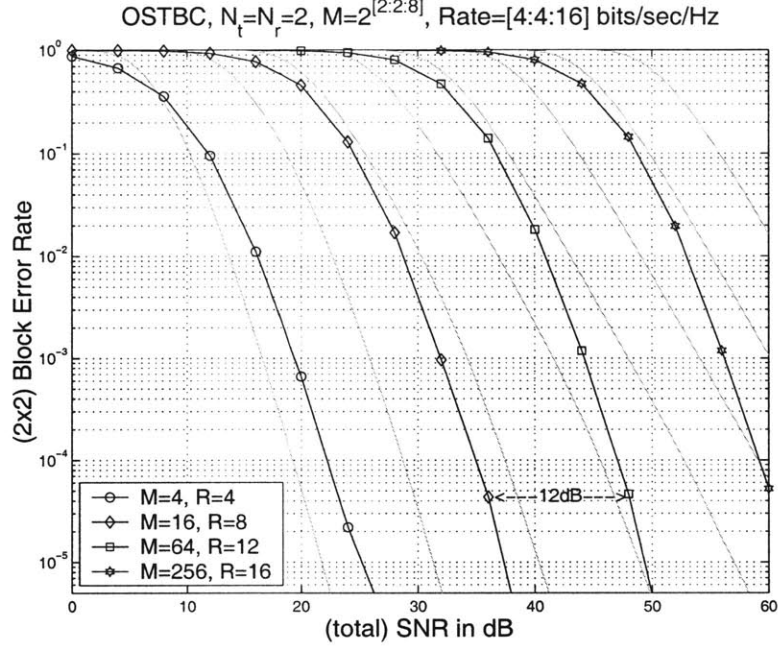


Figure 4-2: Error rate curves of OSTBC (dark) and outage probability curves (light) for various rates. We see that the maximum diversity of four is achieved, but there is a loss of multiplexing gain.

## 4.3 Tilted-QAM Code

### 4.3.1 The Rotation Design

In the OSTBC design we studied in the last section, the key feature allowing it to achieve full diversity is that both information symbols  $s_1$  and  $s_2$  appear in both rows and columns of the codeword matrix  $\mathbf{X}$  via repetition. However, the simple repetition causes a loss of multiplexing gain. In a  $2 \times 2$  codeword matrix, which has four entries, there are effectively only two information symbols.

We propose a new design named *tilted-QAM*, which replaces the repetition in OSTBC with a suitably chosen rotation. For a given transmission rate  $R = r \log_2(\text{SNR})$ , we use a  $M^2$ -QAM constellation carved from  $\mathbb{Z} + \mathbb{Z}j$  with size  $M^2 = 2^{R/2} = \text{SNR}^{r/2}$ . Then, four information symbols,  $s_{ij}$ , instead of two, are encoded into a codeword

matrix  $\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$  via two rotations,

$$\begin{bmatrix} x_{11} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \end{bmatrix},$$

$$\begin{bmatrix} x_{21} \\ x_{12} \end{bmatrix} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} s_{21} \\ s_{12} \end{bmatrix}.$$
(4.5)

To get some intuition of this rotation idea, let us focus on the rotation of  $s_{11}$  and  $s_{22}$  to obtain  $x_{11}$  and  $x_{22}$  as shown in Figure 4-3. The key criterion is that all points except the origin stay off the  $x_{11}$  and  $x_{22}$  axes. In this way, each non-zero information symbol pair  $(s_{11}, s_{22})$  leads to both non-zero  $x_{11}$  and non-zero  $x_{22}$  and effectively appear in both rows and columns of the codeword matrix  $\mathbf{X}$ . With rotation instead of repetition, two  $s_{ij}$  symbols become two  $x_{ij}$  symbols, so there is no sacrifice of multiplexing gain.

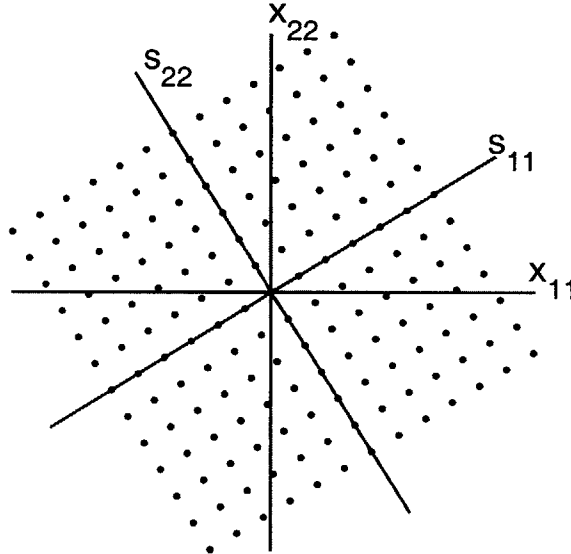


Figure 4-3: Rotate  $(s_{11}, s_{22})$  to obtain  $(x_{11}, x_{22})$ , so that each non-zero information symbol pair  $(s_{11}, s_{22})$  leads to both non-zero  $x_{11}$  and non-zero  $x_{22}$  and effectively appear in both rows and columns of the codeword matrix  $\mathbf{X}$ .

One thing to note is that although it is possible to choose the rotation angle  $\theta_1$  so that all points except the origin stay off the axes as shown in Figure 4-3, it is not possible to keep them a constant distance away from the axes as the constellation grows. This is because if we project such a two-dimensional lattice on to the  $x_{11}$ -axis, it can be shown that the resulting set of points must be dense on the axis. Therefore, there must be points with  $x_{11}$  (and similarly  $x_{22}$ ) arbitrarily close to zero. Interestingly, it turns out that the product  $x_{11}x_{22}$  can be kept a constant distance away from zero, which eventually leads to a certain minimum determinant. In the case of OSTBC, there is essentially a one dimensional lattice along the  $x_{11} = \pm x_{22}$  direction, and  $x_{11}$  and  $x_{22}$  are both kept away from zero by a fixed amount. This is sufficient for having a certain minimum determinant but not necessary.

### 4.3.2 Choice of rotation angles

In this section, using the criterion of maximizing the worst case determinant, we identify a set of rotation angles that is universally optimal and leads to the same worst case determinant for all rates.

While the rotation avoids the multiplexing gain penalty, to ensure maximum diversity (when  $r = 0$ ), we must make all non-zero codeword matrices (equivalent to all difference matrices) full rank. A slightly stronger condition is to maximize the worst case determinant, as discussed in section 2.4. Let the worst case determinant be  $\gamma \stackrel{\text{def}}{=} \min_{\mathbf{X} \neq \mathbf{0}} |\det(\mathbf{X})|$ . We need to choose the two rotation angles to maximize  $\gamma$ .

Let us first look at  $\det(\mathbf{X})$  as a function of  $(\theta_1, \theta_2)$ .

$$\begin{aligned} 2 \det(\mathbf{X}) = & \sin(2\theta_1)(s_{11}^2 - s_{22}^2) + 2 \cos(2\theta_1)s_{11}s_{22} \\ & - \sin(2\theta_2)(s_{21}^2 - s_{12}^2) - 2 \cos(2\theta_2)s_{12}s_{21}. \end{aligned} \quad (4.6)$$

In the case of binary constellation,  $s_{ij}$  each take the value of 0 and 1, so there are only  $2^4 - 1 = 15$  non-trivial 4-tuples. Since sin and cos are both smooth functions, we can easily analytically solve for or search for the best pairs of  $(2\theta_1, 2\theta_2)$  that maximize  $\gamma$ . To demonstrate this visually, we sweep  $2\theta_1$  and  $2\theta_2$  each from 0 to  $\pi$



at 0.02 increment, and plot the resulting minimum determinant as a two-dimensional contour plot in Figure 4-4.

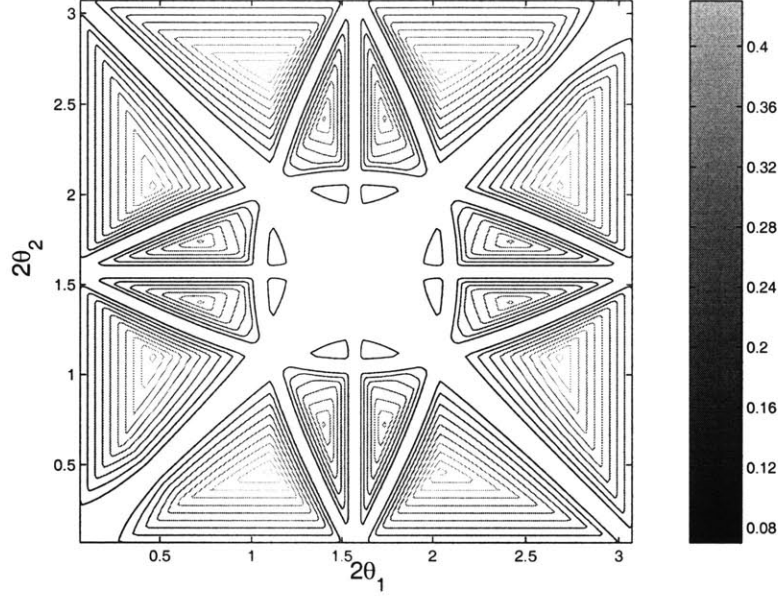


Figure 4-4: Maximize the minimum  $|2 \det(\mathbf{X})|$  as a function of  $2\theta_1$  and  $2\theta_2$  for the case where  $s_{ij}$  each takes the value of 0 and 1.

By solving for the points where the peaks occur, we obtain that one of the optimal choices of  $(2\theta_1, 2\theta_2)$  is  $(\arctan(1/2), \arctan(2))$ . The others are its symmetric variations. With this pair of angles, we evaluate the worst case determinant to be  $\gamma = 1/(2\sqrt{5})$ , and is obtained at, for example,  $(s_{11}, s_{12}, s_{21}, s_{22}) = (1, 0, 0, 0)$ .

In the high rate, high SNR limit, in order to study the diversity-multiplexing tradeoff we need to know explicitly how the optimal angles depend on rate at arbitrarily high rates. This precludes a brute force search for the optimal angles for each rate, which is a method used in many existing literatures [8, 31].

Interestingly, we find that the pair of rotation angles optimal in the binary constellation case is also optimal QAM-like constellations of all sizes. Thus, we have a universal design that maximizes  $\gamma$  for all rates. This result is summarized in the following theorem.

**Theorem 1** For codeword matrix  $\mathbf{X}$  defined in (4.5), the maximum worst case determinant of difference matrices is

$$\max_{(\theta_1, \theta_2)} \min_{\mathbf{X}_1 \neq \mathbf{X}_2} |\det(\mathbf{X}_1 - \mathbf{X}_2)| = \frac{1}{2\sqrt{5}}, \quad (4.7)$$

and achieved by  $(\hat{\theta}_1, \hat{\theta}_2) = (\frac{1}{2} \arctan(\frac{1}{2}), \frac{1}{2} \arctan(2))$  for QAM-like constellations of all sizes.

*Proof:*

For binary constellation, by listing all  $\det(\mathbf{X})$  expressions for all  $s_{ij}$  4-tuples, we showed that  $(\hat{\theta}_1, \hat{\theta}_2)$  and its symmetric variations are optimal, with which  $\gamma = 1/(2\sqrt{5})$  and is obtained at, for example,  $(s_{11}, s_{12}, s_{21}, s_{22}) = (1, 0, 0, 0)$ . As constellation grows,  $\gamma$  can only decrease or remain constant, since there are additional codewords to minimize over. So to prove Theorem 1, it suffices to show that  $\gamma = 1/(2\sqrt{5})$  is actually achievable for larger constellations using  $(\hat{\theta}_1, \hat{\theta}_2)$ , i.e.,  $|\det(\mathbf{X})| \geq 1/(2\sqrt{5})$  for all non-zero 4-tuples of  $s_{ij} \in \mathbb{Z} + \mathbb{Z}j$ .

Substituting  $(\hat{\theta}_1, \hat{\theta}_2)$  into (4.6), we have,

$$J \stackrel{\text{def}}{=} 2\sqrt{5} \det(\mathbf{X}) = s_{11}^2 - s_{22}^2 + 4s_{11}s_{22} + 2s_{12}^2 - 2s_{21}^2 - 2s_{21}s_{12}. \quad (4.8)$$

Since  $s_{ij} \in \mathbb{Z} + \mathbb{Z}j$ , so is  $J$ . Now we need to prove the following.

**Lemma 4.1** For  $s_{ij} \in \mathbb{Z} + \mathbb{Z}j$ ,  $J = s_{11}^2 - s_{22}^2 + 4s_{11}s_{22} + 2s_{12}^2 - 2s_{21}^2 - 2s_{21}s_{12} = 0$  if and only if  $s_{11} = s_{12} = s_{21} = s_{22} = 0$ .

Let us perform completion of squares and change of variables. Let  $a \stackrel{\text{def}}{=} s_{11} + 2s_{22}$ ,  $b \stackrel{\text{def}}{=} s_{22}$ ,  $c \stackrel{\text{def}}{=} 2s_{12} - s_{21}$ , and  $d \stackrel{\text{def}}{=} s_{21}$ , then  $2J = 2a^2 - 10b^2 + c^2 - 5d^2$ . Now we need to prove  $2a^2 + c^2 = 5(2b^2 + d^2)$  only when  $a = b = c = d = 0$ , which requires the following lemma.

**Lemma 4.2** For  $x, y \in \mathbb{Z} + \mathbb{Z}j$ , if  $5|2x^2 + y^2$ , then  $5|x$ ,  $5|y$ , and  $25|2x^2 + y^2$ .<sup>2</sup>

*Proof:*

Let  $x = 5q_x + r_x$  and  $y = 5q_y + r_y$ , such that,  $r_x, r_y \in \{0, 1, 2, 3, 4\} + \{0, 1, 2, 3, 4\}j$  and  $q_x, q_y \in \mathbb{Z} + \mathbb{Z}j$ .  $5|2x^2 + y^2$  implies  $5|2r_x^2 + r_y^2$ . It is straight forward to verify that the only case where  $5|2r_x^2 + r_y^2$  is  $r_x = r_y = 0$ . Therefore,  $5|x$ ,  $5|y$ , and  $25|2x^2 + y^2$ . ■

Now using Lemma 4.2, we can show that

$$\begin{aligned} 2a^2 + c^2 &= 5(2b^2 + d^2) \\ \Rightarrow 5|2a^2 + c^2 &\Rightarrow 5|a, 5|c, 25|2a^2 + c^2 \\ \Rightarrow 5|2b^2 + d^2 &\Rightarrow 5|b, 5|d, 25|2b^2 + d^2 \end{aligned} \quad (4.9)$$

---

<sup>2</sup>For complex integers, divisibility by a real integer (denoted by  $|$ ) is defined as both real and imaginary parts being divisible.

Since all  $a$ ,  $b$ ,  $c$ , and  $d$  are divisible by 5, we can divide both sides of (4.9) by  $5^2$  and obtain an essentially identical equation,  $2a'^2 + c'^2 = 5(2b'^2 + d'^2)$ , where  $a', b', c', d' \in \mathbb{Z} + \mathbb{Z}j$ . We can repeat the above argument and divide both sides by  $5^2$  indefinitely. Thus, the only possible solution is  $a = b = c = d = 0$ , i.e.,  $s_{11} = s_{12} = s_{21} = s_{22} = 0$ . This concludes the proof of Lemma 4.1 and Theorem 1. ■

One follow-up question is how sensitive the worst case determinant is to the values of  $\theta_1$  and  $\theta_2$ . From Figure 4-4, we can see the sensitivity in the case of binary constellation. For larger constellations, the sensitivity in  $\theta_1$  and  $\theta_2$  increases. This is because for larger constellations, a small change in rotation angle can cause the points at the edge of the constellation to move by a larger amount.

We numerically demonstrate this effect. To simplify the computation,  $\theta_1$  is swept from 0 to  $\pi/8$ , while  $\theta_2 = \pi/4 - \theta_1$ . We plot the worst case determinant as a function of  $\theta_1$  for 2-PAM, 3-PAM, 4-PAM, and 5-PAM constellations in Figure 4-5. We can clearly see that the sensitivity of the worst case determinant in terms of  $\theta_1$  increases as constellation gets larger. While the peak is always at  $\theta_1 = \arctan(1/2)/2 = 0.23182$ , it gets sharper and sharper. Although the sensitivity increases with constellation size, for practical constellation sizes like 16-QAM or 64-QAM, the numerical accuracy of the current computers should be sufficient.

## 4.4 Theoretical Performance Analysis

In this section, we analyze the performance of the tilted-QAM code we proposed in the last section. The key property of the tilted-QAM code is that the worst case determinant remains constant as constellation size and rate grows. We show that this determinant property built into the code allows it to achieve the optimal diversity-multiplexing tradeoff for two-transmit two-receive antenna systems.

To evaluate the average error probability of the system, we need to average over both the random channel and the ensemble of codewords. In the next two sections, we present two perspectives. The first one focuses on the error probability of a particular channel averaged over all codewords. We show that when the channel is not in outage, our system tends to have large distances between the received codewords, and thus,

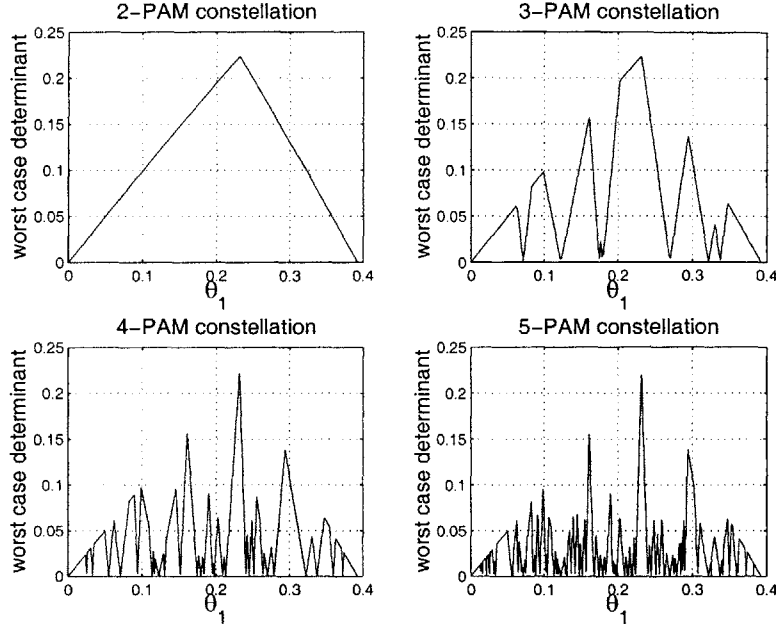


Figure 4-5: Worst-case determinant as a function of  $\theta_1$ , while  $\theta_2 = \pi/4 - \theta_1$ . As constellation size increases, although the optimal value of  $\theta_1$  remains at  $\arctan(1/2)/2$ , the sensitivity increases. Slight deviation of  $\theta_1$  from its optimal value significantly reduces the resulting worst-case determinant.

good performance. The second one first looks at the error probability associated with a particular pair of codewords averaged over all channels, and then sums over the codewords. This perspective allows us to see that the codeword pairs whose differences have small determinants are bad and dominate the overall error probability. Thus, having good determinant property is essential for good performance. We also extend the second perspective to higher dimensional cases in section 4.4.3.

Before the detailed error probability evaluation, let us first express several key parameters as functions of SNR in exponential forms. We need to know how they grow or decay with SNR as SNR grows, because we want to evaluate diversity and multiplexing gains, which are how fast error probability decays and rate grows with SNR.

First of all, we chosen to restrict  $s_{ij} \in \mathbb{Z} + \mathbb{Z}j$ . This means that as SNR increases and the constellation size grows, the separation between the QAM constellation points remains at unity.

With  $M$  denoting the constellation size per dimension, the average transmit energy per dimension  $E_s$  grows with  $M$ ,

$$E_s \doteq M^2 = 2^{R/2} = \text{SNR}^{r/2}. \quad (4.10)$$

The noise level can be expressed as <sup>3</sup>

$$\sigma_w^2 \doteq \frac{E_s}{\text{SNR}} \doteq \text{SNR}^{r/2-1}. \quad (4.11)$$

Let  $\lambda_1 \geq \lambda_2$  denote the singular values of  $\Delta$ . Using the determinant property build into the tilted-QAM code that the worst case determinant is always bounded away from zero by  $1/(2\sqrt{5})$ , we have the lower bound

$$\lambda_1^2 \lambda_2^2 = |\det(\Delta)|^2 \geq \left( \frac{1}{2\sqrt{5}} \right)^2 \doteq \text{SNR}^0. \quad (4.12)$$

The entries of  $\Delta$  are at most order  $M$ . Therefore, we also have an upper bound

$$\lambda_1^2 + \lambda_2^2 = \|\Delta\|^2 \leq M^2 \doteq \text{SNR}^{r/2}. \quad (4.13)$$

Combining the upper and lower bounds, and using  $\lambda_1 \geq \lambda_2$ , we have:

$$\text{SNR}^0 \leq \lambda_1^2 \leq \text{SNR}^{r/2} \quad (4.14)$$

$$\text{SNR}^{-r/2} \leq \lambda_2^2 \leq \text{SNR}^{r/2}. \quad (4.15)$$

These describe how the singular values of  $\Delta$  change with SNR.

#### 4.4.1 Minimum Distance Property

In this section, we evaluate the performance of the tilted-QAM design by studying the minimum distance between received constellation points given a particular channel

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<sup>3</sup>When at the maximum multiplexing gain ( $r = 2$ ), noise variance is fixed. Since the separation between the constellation points is also fixed, the performance remains approximately constant, i.e.,  $d = 0$ .

realization. For a given  $\mathbf{H}$ , the distance between a pair of codewords with difference matrix  $\Delta$  is  $\|\mathbf{H}\Delta\|$ , where the norm  $\|\cdot\|$  is defined as  $\|\mathbf{A}\|^2 = \sum_{i,j} \|a_{ij}\|^2$ . If  $\|\mathbf{H}\Delta\|$  is at least a certain value,  $\delta(\mathbf{H})$ , for all  $\Delta \neq \mathbf{0}$ , then all the received constellation points are at least distance  $\delta$  apart. For a given  $\delta$ , a minimum distance decoder can guarantee to decode correctly when the magnitude of the noise is less than  $\delta/2$ .

To show that the optimal tradeoff can be achieved, we first identify  $\delta(\mathbf{H})$  as a function of  $|\det(\mathbf{H})|$  and  $\|\mathbf{H}\|^2$ . We then relate two expressions, the ratio of  $\delta(\mathbf{H})$  to the noise level,  $\delta^2(\mathbf{H})/\sigma_w^2$ , and the ratio of the realized channel capacity to rate,  $2^{C(\mathbf{H})}/2^R$ . We show that when the channel is not in outage, our system tends to have large distances between codewords and good performance. Finally, we compare the conditional error probability  $P[\text{error}|\mathbf{H}]$  achieved by our code to that of the Gaussian random code and conclude that the tilted-QAM code can achieve the optimal diversity-multiplexing tradeoff.

To lower bound  $\|\mathbf{H}\Delta\|^2$  using  $|\det(\mathbf{H})|$ , we use the minimum determinant property.

$$\|\mathbf{H}\Delta\|^2 \geq 2|\det(\mathbf{H}\Delta)| \geq |\det(\mathbf{H})|. \quad (4.16)$$

To lower bound  $\|\mathbf{H}\Delta\|^2$  using  $\|\mathbf{H}\|^2$ , we note that when multiplied by  $\Delta$ ,  $\mathbf{H}$  must be scaled by at least  $\lambda_2$ , the smaller singular value of  $\Delta$ .

$$\|\mathbf{H}\Delta\|^2 \geq \lambda_2^2 \|\mathbf{H}\|^2 \geq \text{SNR}^{-r/2} \|\mathbf{H}\|^2. \quad (4.17)$$

Combine the above two bounds on  $\|\mathbf{H}\Delta\|^2$  and the noise variance expression (4.11), we can lower bound  $\delta^2(\mathbf{H})/\sigma_w^2$  with  $|\det(\mathbf{H})|$  and  $\|\mathbf{H}\|^2$ ,

$$\tau \stackrel{\text{def}}{=} \frac{\delta^2(\mathbf{H})}{\sigma_w^2} \geq \max \left( \text{SNR}^{1-r/2} |\det(\mathbf{H})|, \text{SNR}^{1-r} \|\mathbf{H}\|^2 \right). \quad (4.18)$$

Let us now relate the channel capacity achieved (2.4) to the quantities  $|\det(\mathbf{H})|$

and  $\|\mathbf{H}\|^2$ . Using  $2^R = \text{SNR}^r$ , we can rewrite (2.4) as

$$\frac{2^{C(\mathbf{H})}}{2^R} \doteq \left( \text{SNR}^{1-r/2} |\det(\mathbf{H})| \right)^2 + \text{SNR}^{1-r} \|\mathbf{H}\|^2. \quad (4.19)$$

Comparing (4.18) and (4.19), we see that both right hand sides involve  $|\det(\mathbf{H})|$  and  $\|\mathbf{H}\|^2$ . When  $C(\mathbf{H})$  is large compared to  $R$ , one of  $|\det(\mathbf{H})|$  and  $\|\mathbf{H}\|^2$  must be large. Consequently,  $\delta^2(\mathbf{H})$  is large compared to  $\sigma_w^2$ . So we have

$$C(\mathbf{H}) > R \implies \tau = \frac{\delta^2(\mathbf{H})}{\sigma_w^2} \geq 1. \quad (4.20)$$

Therefore, when the channel is not in outage, all codewords are well separated compared to the noise level, and correct decoding can be done with high probability. In other words, the error probability achievable by tilted-QAM codes is very close to the channel outage probability. This indicates that the tilted-QAM code should be able to achieve the optimal diversity-multiplexing tradeoff.

Let us take this argument further by examining the conditional error probability  $P[\text{error}|\mathbf{H}]$  achieved as a result of having a large  $\tau = \delta^2(\mathbf{H})/\sigma_w^2$ . We first manipulate the lower bound of  $\tau$  into an exponential form in SNR, and then express  $P[\text{error}|\mathbf{H}]$  in terms of  $\tau$ .

Let  $\lambda_i^{\mathbf{H}}$  be the ordered singular values of  $\mathbf{H}$  and let  $\text{SNR}^{-\alpha_i} = |\lambda_i^{\mathbf{H}}|^2$ , as we did in section 2.3. Then,

$$|\det(\mathbf{H})| = |\lambda_1^{\mathbf{H}} \lambda_2^{\mathbf{H}}| \quad \text{and} \quad \|\mathbf{H}\|^2 = |\lambda_1^{\mathbf{H}}|^2 + |\lambda_2^{\mathbf{H}}|^2.$$

When the channel is not in outage,  $\tau \geq 1$ , we also have  $\tau^2 \geq \tau$ . Equation (4.18) then becomes,

$$\begin{aligned} \tau^2 &\geq \max \left( \text{SNR}^{2-r} |\lambda_1^{\mathbf{H}}|^2 |\lambda_2^{\mathbf{H}}|^2, \text{SNR}^{1-r} (|\lambda_1^{\mathbf{H}}|^2 + |\lambda_2^{\mathbf{H}}|^2) \right) \\ &\geq \text{SNR}^{2-r} |\lambda_1^{\mathbf{H}}|^2 |\lambda_2^{\mathbf{H}}|^2 + \text{SNR}^{1-r} (|\lambda_1^{\mathbf{H}}|^2 + |\lambda_2^{\mathbf{H}}|^2) + \text{SNR}^{-r} \\ &= \text{SNR}^{-r} (\text{SNR} |\lambda_1^{\mathbf{H}}|^2 + 1) (\text{SNR} |\lambda_2^{\mathbf{H}}|^2 + 1) \\ &\doteq \text{SNR}^{\sum_{i=1}^2 (1-\alpha_i)^+ - r} \end{aligned} \quad (4.21)$$

Recall that  $(x)^+$  denotes  $\max(0, x)$ .

Minimum distance decoders can guarantee to decode correctly as long as the magnitude of the noise is smaller than half of  $\delta(\mathbf{H})$ , the minimum distance between codewords. Therefore, from a lower bound on  $\tau = \delta^2(\mathbf{H})/\sigma_w^2$ , we can derive an upper bound on the error probability. Using the fact that the noise magnitude  $\|\mathbf{W}\|^2$  is a chi-squared random variable of order 8, we have,

$$\begin{aligned} P[\text{error}|\mathbf{H}] &< P\left[\frac{\|\mathbf{W}\|^2}{\sigma_w^2} > \tau/2\right] \\ &\doteq \int_{u=\tau/2}^{\infty} u^3 e^{-u} du \\ &\doteq \tau^3 e^{-\tau/2}, \end{aligned} \tag{4.22}$$

where,  $\tau^2 \geq \text{SNR}^{\sum_{i=1}^2 (1-\alpha_i)^+ - r}$  when the channel is not in outage.

Let us compare the  $P[\text{error}|\mathbf{H}]$  achieved to that of the Gaussian random code case. Zheng and Tse showed in [41] that for a Gaussian random code of length  $T$ , when the channel is not in outage, the conditional error probability is

$$P(\text{error}|\mathbf{H}) \leq \text{SNR}^{-T(\sum (1-\alpha_i)^+ - r)}. \tag{4.23}$$

For  $T \geq 3$ , this bound is exponentially tight.

Let  $\eta$  denote  $\text{SNR}^{(\sum (1-\alpha_i)^+ - r)}$  for short. Comparing (4.22) and (4.23), in the latter,  $P(\text{error}|\mathbf{H})$  decays with  $\eta$  like  $\eta^{-T}$ ; in the former,  $P(\text{error}|\mathbf{H})$  decays like  $\eta^{3/2} e^{-\sqrt{\eta}/2}$ . Exponential decays faster than any polynomial, which means that (4.22) behaves like (4.23) with  $T \rightarrow \infty$ . Therefore, the tilted-QAM code has similar performance as a Gaussian random code with infinite code length. Since the latter achieves the optimal diversity-multiplexing tradeoff, so does the tilted-QAM code.

In summary, by looking at the minimum distance properties of the tilted-QAM code, we showed that it can achieve the optimal diversity-multiplexing tradeoff. We note that in order to have this result, we exploit the fact that the worst case determinant remains a constant distance away from zero as rate increases, which is a key property built into the design.



#### 4.4.2 Determinant Counting

We present a different way of evaluating error probability in this section. Earlier, we looked at the performance associated with particular channels. Here, we first look at the error probability associated with a particular pair of codewords averaged over all channels, and then sum over the codewords. While the last method identify what channels are particularly bad, this method allows us to see what codeword pairs dominate the overall error probability.

We first upper bound the pair-wise error probability  $P[\mathbf{X}_1 \rightarrow \mathbf{X}_2]$  by an exponential of SNR. This bound is exponentially tight when  $0 \leq r \leq 1$ , but is loose when  $1 < r \leq 2$ , due to dropping of a “1+” term. Specifically, using (2.21), we have

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \left( \prod_{i=1}^{N_t} \left( 1 + \frac{\lambda_i^2}{\sigma_w^2} \right) \right)^{-2} \leq \left( \prod_{i=1}^{N_t} \left( \lambda_i^2 \text{SNR}^{1-r/2} \right) \right)^{-2} = \frac{1}{|\det(\Delta)|^4} \text{SNR}^{2r-4} \quad (4.24)$$

The above equation is the pair-wise error probability averaged over channel for a particular pair of codewords with difference matrix  $\Delta$ . Notice that the worst kind of codeword pairs are the ones with the smallest determinant, which is order 1. So the worst-pair error probability is  $\text{SNR}^{2r-4}$ . This corresponds to a lower bound on the overall error probability and an upper bound of  $d(r) = 4 - 2r$  on the diversity-multiplexing tradeoff curve. This is a straight line connecting  $(0, 4)$  and  $(2, 0)$ . Comparing to the similar tradeoff curve upper bounds for the Gaussian random codes plotted in Figure 2-10, our upper bound is above that of the length-two expurgated Gaussian random code and is the same as the one with  $T = \infty$ .

In order to obtain the total error probability, we need to use the union bound and sum over all codeword pairs.

$$P_e < \sum_{\Delta \neq 0} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \text{SNR}^{2r-4} \sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^4} \quad (4.25)$$

Recall from (4.8) that  $2\sqrt{5} \det(\Delta) = (s_{11}^2 - s_{22}^2 + 4s_{11}s_{22} + 2s_{12}^2 - 2s_{21}^2 + 2s_{21}s_{12})$ ,

which is a (complex) integer. Let us now look at how often  $2\sqrt{5}\det(\Delta)$  takes on different values. For a constellation of size  $M$ , the range of the determinant is of order  $M^2$ , so there are about  $M^4$  possible complex integer values for  $2\sqrt{5}\det(\Delta)$ . There are order  $M^8$  different  $\Delta$  matrices. So if no value of the determinant is particularly preferred, then the number of  $\Delta$  with a particular determinant should be on the order of  $M^8/M^4 = M^4$ . We can then perform the summation

$$\sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^4} \doteq M^4 \sum_{\substack{-M^4 \leq a, b \leq M^4 \\ (a,b) \neq (0,0)}} \frac{1}{|a + bj|^4} < M^4 \sum_{\substack{-\infty < a, b < \infty \\ (a,b) \neq (0,0)}} \frac{1}{(a^2 + b^2)^2}. \quad (4.26)$$

It can be shown that  $\sum_{\substack{-\infty < a, b < \infty \\ (a,b) \neq (0,0)}} \frac{1}{(a^2 + b^2)^2}$  is a finite constant by using a continuous integral as an upper bound. Consider the piece-wise constant function  $g(x, y)$  that takes the value  $\frac{1}{(a^2 + b^2)^2}$  in the unit square  $[a - 0.5, a + 0.5) \times [b - 0.5, b + 0.5)$ ,  $(a, b) \neq (0, 0)$ . The integral of this function in the domain outside of the unit square around the origin equals the sum we wish to bound. Upper bounding  $g(x, y)$  with  $\frac{100}{(x^2 + y^2)^2}$  and extending the area of integral to  $\sqrt{x^2 + y^2} > 0.5$ , we can upper bound the sum with  $\int_{0.5}^{\infty} \frac{100}{r^4} 2\pi r dr$ , which is clearly some constant. The sum can also be numerically evaluated to be 6.0268.

Having  $\sum_{\substack{-\infty < a, b < \infty \\ (a,b) \neq (0,0)}} \frac{1}{(a^2 + b^2)^2}$  being a constant gives us

$$\sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^4} \doteq M^4 \doteq \text{SNR}^r. \quad (4.27)$$

This implies that we could have just focus on the  $M^4$  difference matrices with the smallest determinant and ignore the rest.

Combining (4.25) and (4.27), we have

$$P_e \doteq \text{SNR}^{2r-4} \text{SNR}^r \doteq \text{SNR}^{3r-4} \quad (4.28)$$

This corresponds to a diversity-multiplexing tradeoff of  $d(r) = 4 - 3r$ , which agrees with the optimal tradeoff for  $0 \leq r \leq 1$ . This shows that the proposed tilted-QAM scheme can achieve the optimal diversity-multiplexing tradeoff for  $0 \leq r \leq 1$ .

We note that the above is not a complete proof because of the step where we argued that there are about  $M^4$  difference matrices  $\Delta$  with a particular determinant. To argue this tightly, it is necessary and sufficient to prove that for  $-M \leq \text{Re}(s_{ij}) \leq M$ ,  $-M \leq \text{Im}(s_{ij}) \leq M$ , and any  $J \in \mathbb{Z} + \mathbb{Z}j$ ,  $(s_{11}^2 - s_{22}^2 + 4s_{11}s_{22} + 2s_{12}^2 - 2s_{21}^2 + 2s_{21}s_{12}) = J$  has at most order  $M^4$  solutions.

At this point, it is still a conjecture without proof. The argument above simply seemed reasonable and agrees with our numerical simulations in which the number of solutions is counted. In our simulation, we count the number of times  $|J(s_{ij})| = 1$ , with  $s_{ij}$  taking only real values between  $-M$  and  $M$  instead of complex numbers to allow ourselves to go to greater  $M$ . With real numbers, we expect the number of solutions to be of order  $M^2$ . We grow  $M$  exponentially from 4 up to 256 at increments of around  $\sqrt{2}$ . The number of solutions as a function of  $M$  is plotted in Figure 4-6 on a log-log scale so that exponent is revealed as slope. We see that the curve approaches a straight line as  $M$  becomes sufficiently large. A linear fit of the curve from  $M = 16$  to  $M = 256$  shows a slope of 1.97 and another linear fit of the curve from  $M = 64$  to  $M = 256$  shows a slope of 1.99. This numerical evidence shows that the number of solutions seems to grow like  $M^2$  for the real case (and  $M^4$ , for complex). Further work is still needed to formally establish this result.

Although this determinant counting perspective does not provide a proof for the optimality of the tilted-QAM code as the minimum-distance perspective did in the last section, it nevertheless provides an intuition for what contributes the most to error events and how worst case determinant plays a role. It is interesting to note that in the tilted-QAM code, there are many worst-case codeword pairs. This could be interpreted as that the codewords are so carefully placed that they are equally close to many other codewords in many directions.

#### 4.4.3 Determinant Counting: Higher Dimensional Cases

In the previous sections, we have focused on a multiple antenna system with two-transmit two-receive antennas. The optimal diversity multiplexing tradeoff curve has two piece-wise linear segments, between  $0 \leq r \leq 1$  and  $1 \leq r \leq 2$ . Generally speaking,

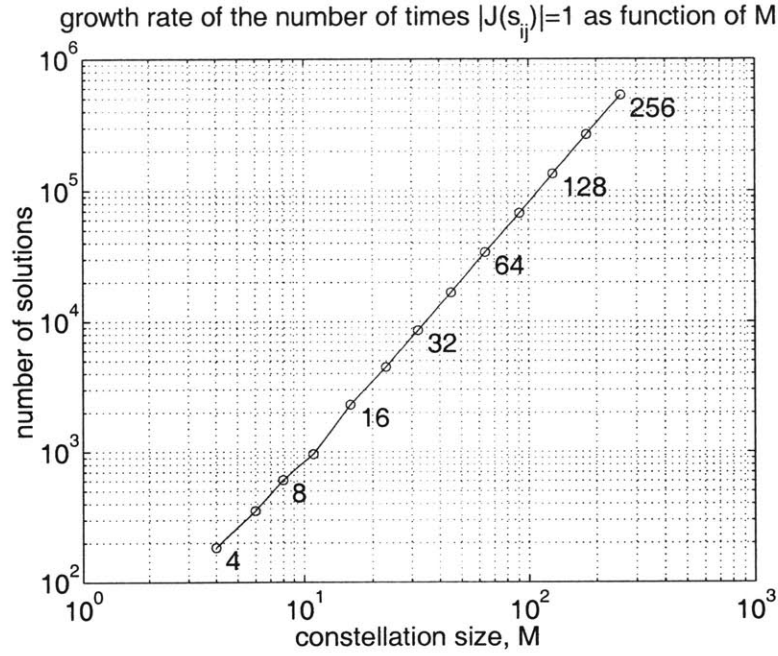


Figure 4-6: Growth rate of the number of matrices with a particular determinant as a function of the constellation size  $M$ .

for a multiple antenna system with  $N_t$  transmit antennas and  $N_r$  receive antennas, the optimal diversity multiplexing tradeoff curve has  $K = \min(N_t, N_r)$  pieces, as shown in Figure. 2-3. In this section, based on our experience in the  $2 \times 2$  case, let us speculate how performance evaluation may be done for higher dimensional cases, in particular, for the segment with  $0 \leq r \leq 1$ , using the determinant counting method described in section 4.4.2.

Using this technique, we can show that for the  $T = N_t$  case, if a design could guarantee a worst-case determinant of order 1 and does not involve repetition, then it would achieve the optimal diversity-multiplexing tradeoff for  $0 \leq r \leq 1$ . We briefly walk through the reasoning next.

First we have

$$M^2 \doteq \text{SNR}^{r/N_t} \quad \text{and} \quad \sigma_w^2 \doteq \text{SNR}^{r/N_t - 1}. \quad (4.29)$$

The pair-wise error probability averaged over all channels is

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \leq \left( \prod_{i=1}^{N_t} \left( 1 + \frac{\lambda_i^2}{\sigma_w^2} \right) \right)^{-N_r} = \frac{1}{|\det(\Delta)|^{2N_r}} \text{SNR}^{rN_r - N_t N_r} \quad (4.30)$$

The total error probability can be upper bounded using the union bound :

$$P_e < \sum_{\Delta \neq 0} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \text{SNR}^{rN_r - N_t N_r} \sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^{2N_r}} \quad (4.31)$$

Again, we need to count the number of times each  $\det(\Delta)$  value occurs. There are  $M^{2N_t^2}$  codewords, and the range of  $\det(\Delta)$  is of order  $M^{N_t}$ . So each determinant occurs about  $M^{2N_t^2 - 2N_t} = M^{2N_t(N_t - 1)} = \text{SNR}^{r(N_t - 1)}$  times. Focusing only on those with the smallest determinant, the overall error probability is

$$P_e \doteq \text{SNR}^{rN_r - N_t N_r} \text{SNR}^{r(N_t - 1)} = \text{SNR}^{-(N_t N_r - rN_r - rN_t + 1)}.$$

Therefore, the diversity-multiplexing tradeoff achieved is  $d(r) = N_t N_r - rN_r - rN_t + 1$ . Evaluating it at  $r = 0$  and  $1$ , we have  $d(0) = N_t N_r$  and  $d(1) = (N_t - 1)(N_r - 1)$ . Therefore, the tradeoff achieved agrees with the optimal tradeoff in Lemma 2.1 for  $0 \leq r \leq 1$ .

This tells us that for the  $T = N_t$  case, if we could design a codebook without using repetition and guarantee that the smallest determinant is of order 1, then it would achieve the optimal diversity-multiplexing tradeoff curve for  $0 \leq r \leq 1$ . Also, since the code takes in  $N_t^2$  information symbols, like the tilted-QAM design instead of the OSTBC design, we expect the code to achieve the  $(N_t, 0)$  points. (Assume  $N_r \geq N_t$ , so we are not losing any dimensions.) At this point, it is unclear whether these two properties are sufficient for achieving all the intermediate tradeoff points. We suspect that other criteria such as maximizing the minimum of some other functions of  $\Delta$ , not just the determinant, might be needed.

## 4.5 Simulation Results

In this section, we use numerical simulations to verify that the tilted-QAM code we proposed in section 4.3 can indeed achieve the optimal diversity-multiplexing tradeoff as our theoretical analysis in section 4.4 suggests.

We generate a family of block error rate curves for various rates and compare them to outage probability curves shown in Figure 2-5. We demonstrate that our block error rate curves exhibit similar characteristics as the outage probability curves, which indicates that they have similar diversity-multiplexing tradeoffs. We also show that tilted-QAM code significantly out-performs OSTBC in the high SNR regime. Finally, we explore the possibility of using the lower complexity lattice-reduction based decoding introduced in chapter 3, instead of using the more-complex maximum likelihood decoding. We show that, with tilted-QAM code, lattice decoding is sub-optimal and results in similar performance as an uncoded system.

For the tilted-QAM coding scheme, four information symbols  $s_{ij}$  chosen out of QAM-like constellations are encoded into a  $2 \times 2$  transmitted signal matrix  $\mathbf{X}$  according to (4.5). The matrix  $\mathbf{X}$  is then transmitted over the multiple antenna channel,  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ . Random channels with IID  $\mathcal{CN}(0, 1)$  entries are generated for each trial. At the receiver, we must deal with the combined effect of the encoder and the channel. We write the received signal  $y_{ij}$  directly in terms of the information symbols  $s_{ij}$  as

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & & \\ h_{21} & h_{22} & & \\ & & h_{11} & h_{12} \\ & & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{bmatrix} \begin{bmatrix} c_1 & -s_1 & & \\ s_1 & c_1 & & \\ & & c_2 & -s_2 \\ & & s_2 & c_2 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{21} \\ s_{12} \end{bmatrix} + \begin{bmatrix} w_{11} \\ w_{21} \\ w_{12} \\ w_{22} \end{bmatrix}, \quad (4.32)$$

where  $c_i = \cos(\theta_i)$  and  $s_i = \sin(\theta_i)$ . We can write (4.32) as  $\mathbf{Y}_{\text{vec}} = \mathbf{H}_{\text{eff}}\mathbf{S}_{\text{vec}} + \mathbf{W}_{\text{vec}}$ , where the subscript “vec” indicates vectorized form. Because of this relationship, the received constellation is a skewed version of the original (uncoded) integer con-

stellation and is part of a four-complex-dimensional lattice. Therefore, we can use the sphere decoding technique [26, 38] to reasonably efficiently implement maximum likelihood (ML) or minimum distance decoding, which is what we assumed in the analytical performance evaluation.

#### 4.5.1 ML/Sphere Decoding

We perform simulations using the tilted-QAM encoding scheme and ML decoder at rates  $R = 4, 8, 12, \dots, 32$  b/s/Hz using constellations with sizes per dimension,  $M = 2, 4, 8, \dots, 256$ . We note that  $R = 2 \cdot \log_2(M^2)$ . The resulting family of  $2 \times 2$  block error rate curves for the various rates are plotted in Figure 4-7. The outage probability curves for the those rates are also plotted for comparison.

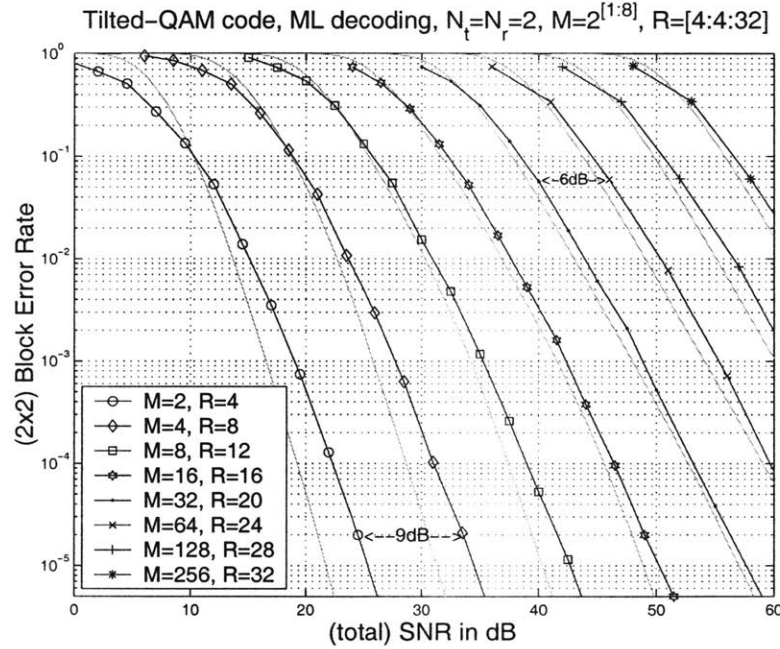


Figure 4-7: Error rate curves of the proposed titled-QAM code (dark) and the outage probability curves (light) for various rates. We see that the two sets of curves have similar slopes and horizontal gaps, which means that they have similar diversity and multiplexing gains.

We see that the tilted-QAM block error rate curves follow the outage probability curves closely, especially at higher rates. At lower rates, the curves do not agree as

well. This is because diversity-multiplexing tradeoff is a high SNR characteristic. It is possible for two systems with the same tradeoff to have different low SNR behavior.

The diversity and multiplexing gains achieved can be measured from the slopes of the error rate curves and the horizontal spacings between these curves, as discussed earlier in section 2.3. Let us compare the slopes and gaps achieved by the tilted-QAM code to that of the linearized outage probability curves show in Figure 2-7. We see that above the  $P_{\text{out}} = \text{SNR}^{-1}$  line, the gaps between the curves with rate differential 4 b/s/Hz is about 6 dB. This implies the maximum multiplexing gain of 2 b/s/Hz per 3 dB. At this location, the slope of the curves is about 2. Below the  $P_{\text{out}} = \text{SNR}^{-1}$  line, the slope of each curve approaches 4, which is the maximum diversity gain. The gaps between the curves is about 9 dB, which corresponds to 4/3 b/s/Hz per 3 dB. All these slopes and gaps agrees with the optimal tradeoff curve in Figure 2-4.

These simulation results show that the proposed tilted-QAM encoding scheme, together with ML decoding, can match the outage probability curves and achieve the optimal diversity-multiplexing tradeoff.

We also note that the diversity-multiplexing tradeoff does not capture constant factor differences between systems. One system may be a fixed dB inferior than another, while having the same tradeoff. Our simulation results show that the gap between the tilted-QAM code and the outage probability is in fact quite small, even though the tilted-QAM code is only designed to achieve the optimal diversity-multiplexing tradeoff.

Comparing the tilted-QAM code and the OSTBC performance show in Figure 4-2, we see that at 4 b/s/Hz, they are similar. For rates below 4 b/s/Hz, OSTBC is near optimal and is preferred for its lower decoding complexity. As rate increases, tilted-QAM codes out-perform OSTBC by increasing amounts due to the superior multiplexing gain. Tilted-QAM codes achieve the same rates at much lower SNR; and since they reach the same limiting slopes, OSTBC never catches up.



## 4.5.2 Lattice Decoding

Earlier in chapter 3, we proposed a lattice-reduction-aided detector that has lower complexity and achieves near ML performance. One draw back is that this decoder treats the constellation as an infinite lattice and does not handle constellation boundaries.

In this section, we investigate the degree to which this low-complexity decoder can replace the more-complex ML decoder when the transmitter uses the tilted-QAM scheme. We show that the maximum diversity can not be achieved due to the boundary problem and the resulting performance is similar to an uncoded system. We first provide intuition and then present numerical simulation results.

Intuitively, we speculate that due to the boundary issue, the lattice decoder can not perform as well as the ML decoder. When the constellation boundary is not considered during decoding, there are effectively many more codeword pairs and many more difference matrices with small determinant. The determinant counting method in section 4.4.2 suggests that this can lead to significant performance degradation.

From a different perspective, without the boundary, there is no upper bound on the energy of the difference matrix  $\|\Delta\|^2$  as in (4.13). Consequently, there is no lower bound on the smaller singular value of  $\Delta$ ,  $\lambda_2$ , as in (4.15). As a result, there can never be an SNR large enough so that the  $(1 + \lambda_2^2/\sigma_w^2)$  term in the pair-wise error probability is in effect. Without the contribution from this  $\lambda_2$  term, the slopes of the error rate curves can only reach 2.

To verify our speculation, we perform simulations with tilted-QAM encoder and lattice-reduction-aided BLAST decoder at the same constellation sizes and rates as before. The results are plotted in Figure 4-8. We see that, as we predicted, the slopes reach a maximum of only 2 and never reach 4. The gaps between the curves are still 6 dB since we do not lose any multiplexing gain.

We notice that Figure 4-8 looks very similar to Figure 3-9, the performance of an uncoded system with lattice reduction aided detector. This means that when using lattice decoding, there is no benefit to using the tilted-QAM code. This is because

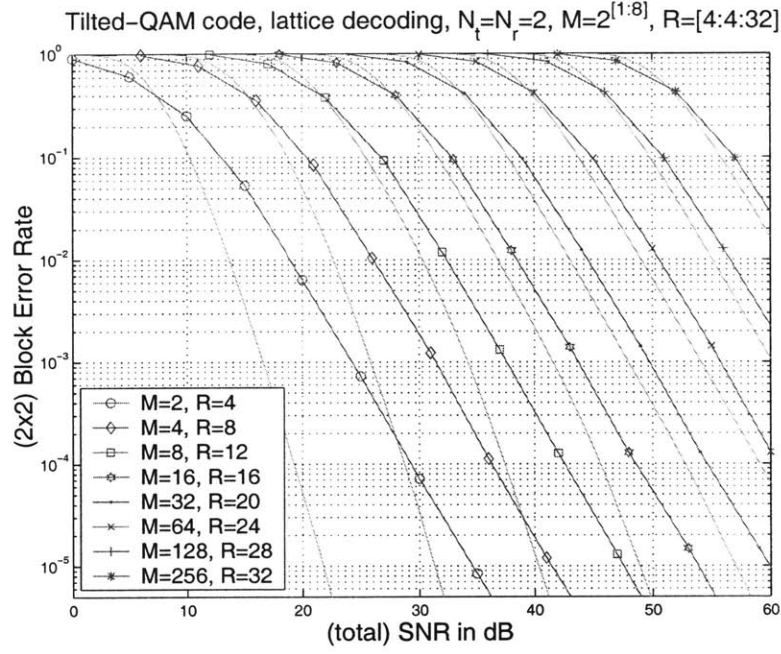


Figure 4-8: Tilted-QAM encoding with lattice-reduction-aided BLAST decoding. The maximum slope reached is only 2. The gaps between the curves are 6 dB, indicating full multiplexing gain.

when the boundary is not handled,  $\lambda_2$  can be arbitrarily small. At any given SNR, a  $\lambda_2$  much smaller than  $\sigma_w$  is equivalent to zero. So the matrix is effectively singular and the determinant equals zero.

In summary, the tilted-QAM code with ML decoding achieves the optimal diversity-multiplexing tradeoff. It out-performs OSTBC due to the superior multiplexing gain. However, if the ML decoder is replaced with a lattice decoder, then the optimal performance is lost. In fact, the tilted-QAM code becomes ineffective, and the performance achieved is similar to that of an uncoded system.

## 4.6 Tilted-QAM in Single Antenna Case

In the past three sections, we described a tilted-QAM code design for the two-transmit two-receive antenna channel. It has a specifically chosen set of universally optimal rotation angles that maximizes the worst case determinant for all rates, and it achieves

the optimal diversity-multiplexing tradeoff.

In this section, we apply similar design techniques to the single antenna fading channel problem. We consider the scenario where we are allowed to code over two independent channel realizations, which resembles coding over two different antennas. We see that this coding problem can be viewed as a simpler version of the previous problem with fewer variables.

We first describe the channel model for this system and present the related capacity and diversity-multiplexing tradeoff results. Next, we show a modified version of the tilted-QAM design for this problem. We then use the determinant counting technique to show that this design achieves its respective optimal tradeoff.<sup>4</sup>

We note here that the code design we propose here was also proposed by Boutros and Viterbo in [3]. The codes are designed using the same determinant criterion. What is new in this work is that we show the universality of the design for all rates, and our focus is on diversity-multiplexing tradeoff. We evaluate the tradeoff achievable by this code and compare it to the optimal tradeoff of the system.

#### 4.6.1 Channel Model and Theoretical Background

The single antenna Rayleigh fading channel with AWGN can be modeled as

$$y = hx + w, \tag{4.33}$$

where  $h$  has zero-mean, unit variance, complex Gaussian density,  $\mathcal{CN}(0, 1)$ ,  $x$  represents the transmitted signal,  $w$  is the AWGN, and  $y$  is the received signal. The average signal to noise ratio is  $\rho$ .

We consider the scenario where we are allowed to code over two independent channel realizations,  $h_1$  and  $h_2$ . In this case, the system model is illustrated in Figure 4-9. Comparing to the multiple antenna channel model in Figure 1-1, this is

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<sup>4</sup>We present an analytical determinant counting argument for the case where all the variables are limited to the real field. For the complex case, due to the additional dimensions involved, it is difficult to handle all the variables.

essentially a two-transmit two-receive system without the cross interference, which only makes the problem easier.

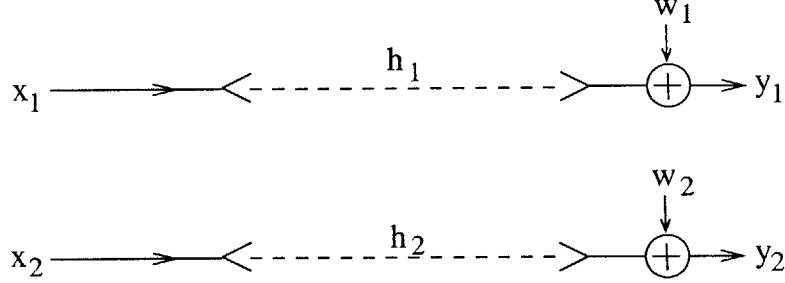


Figure 4-9: Single antenna fading channel over two channel realizations.

The average capacity per channel use achievable by this system is

$$C = \frac{1}{2} (\log_2(1 + \rho|h_1|^2) + \log_2(1 + \rho|h_2|^2)) \quad (4.34)$$

$$= \frac{1}{2} \log_2 (\rho^2|h_1|^2|h_2|^2 + \rho(|h_1|^2 + |h_2|^2) + 1) \quad (4.35)$$

From the capacity expression, we can derive the optimal diversity-multiplex trade-off of this system using the same technique used in section 2.3.3. When the target transmission rate is  $R = r \log_2(\rho)$ , the outage probability is

$$\begin{aligned} P_{\text{out}}(R, \rho) &= P[C < R] \\ &= P[\rho^2|h_1|^2|h_2|^2 + \rho(|h_1|^2 + |h_2|^2) + 1 < \rho^{2r}] \\ &\doteq P[|h_1|^2 < \rho^{r-1} \text{ and } |h_2|^2 < \rho^{r-1}] \\ &\doteq \rho^{r-1} \cdot \rho^{r-1} \\ &= \rho^{2r-2}, \end{aligned} \quad (4.36)$$

where we use the property that  $|h_1|^2$  and  $|h_2|^2$  are chi-squared random variables of order 2. Therefore, the optimal diversity-multiplex tradeoff achievable by this system is  $d(r) = 2r - 2$ , a straight line between  $(0, 2)$  and  $(1, 0)$ .

### 4.6.2 Tilted-QAM design

In this study, we are interested in code designs that achieve the optimal diversity-multiplexing tradeoff. We consider the shortest non-trivial code which consists of two symbols, one going through each of the channel realizations. We can represent this system in a matrix form  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , more specifically,

$$\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} + \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}. \quad (4.37)$$

Since all the matrices are diagonal, we can modify the tilted-QAM design for this single antenna fading channel case by simply using the diagonal terms of a tilted-QAM code.

We propose a design where the codeword matrix  $\mathbf{X} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$  is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (4.38)$$

Again,  $s_i$  are uncoded information symbols chosen independently and uniformly out of a QAM-like constellation carved from  $\mathbb{Z} + \mathbb{Z}j$ .

For any diagonal matrix, its determinant is simply the product of its diagonal elements. Using the same technique used for finding the rotation angle pair for the multiple antenna design in section 4.3.2, we find that the optimal angle that maximizes the worst determinant in this case is

$$\theta = \frac{1}{2} \arctan(2). \quad (4.39)$$

With this choice of rotation angle, the resulting determinant is

$$\det(\mathbf{X}) = x_1 x_2 = \frac{1}{\sqrt{5}} (s_1^2 + s_1 s_2 - s_2^2). \quad (4.40)$$

This determinant is never zero unless both  $s_1$  and  $s_2$  are zero. The proof is a special

case of the proof for Lemma 4.1 with just two of the variables, instead of four.

### 4.6.3 Error Probability Evaluation

Next, let us follow the earlier analysis done for multiple antenna channels and derive error probability expressions for this single antenna channel with the modified tilted-QAM code design. We again use the determinant counting technique used in section 4.4.2.

From the error probability expression derived in section 2.4, we have,

$$\begin{aligned}
P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] &\leq E_{\mathbf{H}} \left[ \exp \left\{ \frac{-\|\mathbf{H}\Delta\|^2}{8\sigma_w^2} \right\} \right] \\
&= E_{h_1} \left[ \exp \left\{ \frac{-|h_1|^2|\delta_1|^2}{8\sigma_w^2} \right\} \right] E_{h_2} \left[ \exp \left\{ \frac{-|h_2|^2|\delta_2|^2}{8\sigma_w^2} \right\} \right] \\
&= \frac{1}{1 + \frac{|\delta_1|^2}{8\sigma_w^2}} \cdot \frac{1}{1 + \frac{|\delta_2|^2}{8\sigma_w^2}},
\end{aligned}$$

where  $\Delta$  is the difference codeword matrix with diagonal elements  $\delta_1$  and  $\delta_2$ .

Mirroring (4.10) and (4.11), we have here,

$$E_s \doteq M^2 = 2^R = \rho^r \quad \text{and} \quad \sigma_w^2 \doteq \frac{E_s}{\rho} \doteq \rho^{r-1}. \quad (4.41)$$

Combining the above, we have, similar to (4.24),

$$P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \frac{1}{1 + \frac{|\delta_1|^2}{8\sigma_w^2}} \cdot \frac{1}{1 + \frac{|\delta_2|^2}{8\sigma_w^2}} \leq \frac{1}{(|\delta_1|^2|\delta_2|^2)} \sigma_w^4 = \frac{1}{|\det(\Delta)|^2} \rho^{2r-2}. \quad (4.42)$$

In order to obtain the total error probability, we need to use the union bound and sum over all codeword pairs, we have, similar to (4.25),

$$P_e < \sum_{\Delta \neq 0} P[\mathbf{X}_1 \rightarrow \mathbf{X}_2] \doteq \rho^{2r-2} \sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^2} \quad (4.43)$$

Now, in order to prove that the optimal tradeoff of  $d(r) = 2r - 2$  is achieved, we need to show that  $\sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|^2}$  grows slower than any polynomial power of  $\rho$  as

rate and constellation size increases.

When all the variables are complex, the summation happens over a four dimensional integer grid, two for each complex diagonal elements of  $\Delta$ . This summation is difficult. Instead, let us change the problem and perform this summation for real variables to develop some intuitions for the complex case.

For the real case,  $|h_1|^2$  and  $|h_2|^2$  are chi-squared random variables of order 1, instead of 2. Because of this, we should sum over  $\frac{1}{|\det(\Delta)|}$  instead of  $\frac{1}{|\det(\Delta)|^2}$ .

From the determinant property of the modified tilted-QAM code in (4.40), we have

$$|\det(\Delta)| = |a^2 + ab - b^2|, \quad (4.44)$$

where  $(a, b)$ ,  $a, b \in \mathbb{Z}$ , represents the difference between two information symbol pairs.

When the constellation size is  $M$ , the range of  $a$  and  $b$  is within  $-M$  and  $+M$ . Now what we need to evaluate is

$$\sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|} = \sum_{\substack{-M < a, b < M \\ (a, b) \neq (0, 0)}} \frac{1}{|a^2 + ab - b^2|}. \quad (4.45)$$

In the following lemma, we show that the right hand side quantity grows no faster than  $(\log M)^2$ . Since  $M^2 = \rho^r$ , this in turn implies that  $\sum_{\Delta \neq 0} \frac{1}{|\det(\Delta)|} \leq \left(\frac{r}{2} \log \rho\right)^2$ , which grows slower than any polynomial power of  $\rho$ .

**Lemma 4.3** For  $a, b \in \mathbb{Z}$ ,

$$\sum_{\substack{-M < a, b < M \\ (a, b) \neq (0, 0)}} \frac{1}{|a^2 + ab - b^2|} \leq (\log M)^2. \quad (4.46)$$

*Proof:*

Let us first divide all the points  $(a, b)$  to be summed over into the standard four quadrants. To take care of the axis, let each quadrant include the semi-axis on its clockwise side. For example, the first quadrant include all points  $a \geq 1, b \geq 1$  and the positive x-axis. Note that no quadrant contains the origin.

Let  $f(a, b) = |a^2 + ab - b^2|^{-1}$ . It has the symmetry property that

$$f(a, b) = f(-a, -b) = f(b, -a) = f(-b, a). \quad (4.47)$$

This means that every point in the second, third, and fourth quadrant has a corresponding image in the first quadrant. Therefore,

$$\sum_{\substack{-M < a, b < M \\ (a, b) \neq (0, 0)}} f(a, b) = 4 \cdot \sum_{\substack{1 \leq a < M \\ 0 \leq b < M}} f(a, b). \quad (4.48)$$

To sum over the first quadrant, we further divide it into two regions,  $b > a$  and  $a \geq b$ . We map points with  $b > a$  to points in the  $a \geq b$  region by using the identity

$$f(a, b) = f(b - a, a). \quad (4.49)$$

For each point  $(a, b)$  with  $b > a \geq 1$ , we map it to  $(b - a, a)$ . Since  $b > a$  to start with,  $b - a \geq 1$ . Therefore, the new point is still inside the first quadrant. We continue with this mapping until we get a point with  $a \geq b$ . For example, starting from  $(11, 17)$ , we first map it to  $(6, 11)$ , then  $(5, 6)$ ,  $(1, 5)$ , and finally  $(4, 1)$ ,  $4 > 1$ .

All these points, eg.,  $(11, 17), \dots (4, 1)$ , have the same  $f(a, b)$  value and they all map to the same point in the  $a \geq b$  region. We need to count how many points in the  $b > a$  region map to the same point in the  $a \geq b$  region. We do so by noticing that this sequence of coordinates is Fibonacci like,  $(b - a, a) \leftarrow (a, (b - a) + a) = (a, b)$ . We know that Fibonacci number grow exponentially (with limiting rate  $\frac{\sqrt{5}+1}{2}$ ). Thus, within a certain range  $M$ , there are  $\log M$  many such points. Therefore, each point in the  $a \geq b$  region is mapped to by at most order  $\log M$  points in the  $b > a$  region.

Now, we can just sum over the  $a \geq b$  region and multiply the result by  $\log M$  to take care of all the points in the  $b > a$  region as an upper bound to the total sum.

To sum over the  $a \geq b$  region in the first quadrant, we use an upper bound of  $f(a, b)$ . When  $a \geq b \geq 0$ ,  $|a^2 + ab - b^2| \geq a^2$ . Thus  $f(a, b) \leq \frac{1}{a^2}$ . Therefore,

$$\sum_{\substack{1 \leq a < M \\ 0 \leq b \leq a}} f(a, b) \leq \sum_{\substack{1 \leq a < M \\ 0 \leq b \leq a}} \frac{1}{a^2} = \sum_{1 \leq a < M} \frac{a+1}{a^2} \leq \sum_{1 \leq a < M} \frac{1}{a} \leq \log M. \quad (4.50)$$

In summary,

$$\sum_{\substack{-M < a, b < M \\ (a, b) \neq (0, 0)}} \frac{1}{|a^2 + ab - b^2|} = 4 \sum_{\substack{1 \leq a < M \\ 0 \leq b < M}} f(a, b) \leq \log M \sum_{\substack{1 \leq a < M \\ 0 \leq b \leq a}} f(a, b) \leq (\log M)^2. \quad (4.51)$$

■

To numerically verify Lemma 4.3, we plot  $\sum_{\substack{-M < a, b < M \\ (a, b) \neq (0, 0)}} f(a, b)$  as a function of  $M$  for  $M$  up to 1000 in Figure 4-10. We also plot the curve  $5(\log M)^2$  on top of it. We see that the sum seems to grow a little slower than  $5(\log M)^2$ .

We also plot the number of times where  $f(a, b) = 1$ , which is a significant part



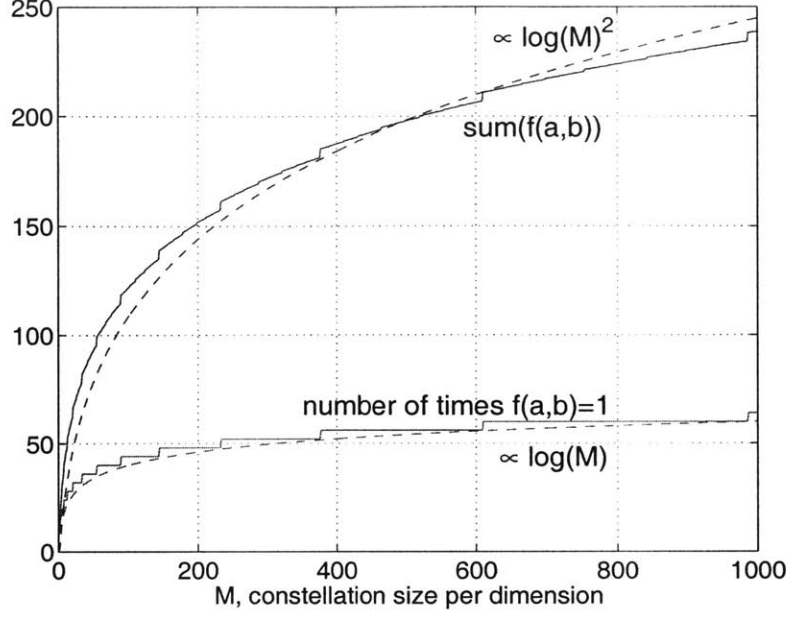


Figure 4-10: The sum  $\sum_{\substack{-M < a, b < M \\ (a,b) \neq (0,0)}} f(a,b)$  as a function of  $M$ , the number of times  $f(a,b) = 1$ , and their approximations (dash).

in the total sum. We see that it grows like  $\log M$ . We can in fact list all of the solutions. They are the symmetric variations (in the four quadrants) of the point  $(1, 0), (1, 1), (1, 2), (2, 3), (3, 5), \dots$ , the well-known Fibonacci sequence.

## 4.7 Summary

In this chapter, we studied the problem of designing structured deterministic codes that achieve the optimal diversity-multiplexing tradeoff. In particular, we focused on the two-transmit two-receive antennas case, and length two codes, the minimum needed to achieve the optimal tradeoff. We reviewed the well-known OSTBC code, which uses a smart repetition to achieve the maximum diversity gain. In doing so, it sacrifices multiplexing gain.

Realizing the problem of OSTBC, we proposed a tilted-QAM coding design which replaces the repetition with a suitably chosen rotation while keeping the cross diagonal structure. Based on the criterion of maximizing the worst case determinant, a set of rotation angles is identified and proven to be universally optimal for all rate. This

universal characterization of the code allows us to analyze its performance in the high SNR regime. It is then shown that the proposed tilted-QAM design achieves the optimal diversity-multiplexing tradeoff through both theoretical analysis as well as numerical simulations.

Prior to this work, there is no known scheme that achieves the optimal diversity-multiplexing tradeoff for the  $N_t = N_r = T = 2$  case. After Zheng and Tse showed that Gaussian random code is sub-optimal in this case, it was left as an open question whether the optimal tradeoff is even achievable at this length. This question is now answered by our work.

The key to our design is the identification of the rotation angles which guarantees that the worst case determinants remain a constant distance away from zero as rate increases.

Comparing tilted-QAM code and OSTBC, similar performance is achieved at 4 b/s/Hz. Above that, tilted-QAM out-performs OSTBC by increasing amounts. At lower rates, OSTBC is preferred for its lower complexity.

# Chapter 5

## Error Correction Code Enhanced Systems

### 5.1 Introduction

In the previous chapter we studied coding for a two-transmit two-receive antenna system with a length two code that can effectively achieve the optimal diversity-multiplexing tradeoff. In this chapter we further investigate the role of using longer, more powerful, error correction codes. The goal is to understand how to build practical systems with good performance.

In communication systems, it is a common practice to introduce redundancy into the transmitted signal via coding to improve performance. Error correction coding for AWGN channels has long been studied. There are well-known soft-decision codes like the turbo codes and LDPC codes that can approach capacity to within a small fraction of a dB. There are also hard-decision codes like Reed-Solomon codes, that have been used in industry for decades.

These codes typically provide coding gains that are measured in terms of constant gains in SNR in dB. This is different from coding for diversity-multiplexing tradeoff for multiple antenna channels, which is about the slopes at which probability of error decays or data rate increases with SNR, rather than constant offsets. Therefore, we must use long error correction code in addition to codes specifically designed for

multiple antenna channels to achieve both good diversity-multiplexing tradeoff and good constant factor gain, and help us obtain the best performance possible.

Another reason for using error correction coding is that, in practice, data is often sent in packets of many hundreds of bytes or longer. What users really care about is the block error rate of such a long block. For example, an executable file must be received completely correctly; even a few bit errors would make the file useless. If we only use the short block codes discussed in the last chapter, a packet would consist of many hundreds of separately coded small blocks. The probability of getting one of them wrong is very high. Therefore, we must use error correction coding to introduce redundancy into the entire block to protect it. In addition, error correction coding provides a mean of error detection, so re-transmission can take place if needed.

In this study, for simplicity, we mainly focus on coding within one channel realization, where the length of the error correction code used is shorter than the channel coherence time, so that only one channel realization is seen by each codeword. When the channel is fast varying and we can afford relatively longer delay, we can consider coding over multiple channel realizations. This would provide additional temporal diversity, because all the channels have to fade simultaneously for the transmission to fail. Coding across channel realizations can always be implemented as a higher level outer code.

The system model we use in this chapter is again  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , where  $\mathbf{X}$  is the  $2 \times T$  transmitted signal matrix with large  $T$ ,  $\mathbf{H}$  is the  $2 \times 2$  channel matrix,  $\mathbf{W}$  is the additive white Gaussian noise and  $\mathbf{Y}$  is the received signal. Under the Rayleigh fading model, the entries of  $\mathbf{H}$  are independent and identically distributed  $\mathcal{CN}(0, 1)$  random variables, and are assumed to be known by the receiver, but not the transmitter.

We study several existing, as well as newly proposed, coding schemes and obtain some understanding of their potential and limitation. We look at what performance they can achieve and discuss their problems.

The outline of this chapter is as follows. First, we briefly look at a system based on the orthogonal space-time block code and show that it is near optimal when operating in the low SNR regime but increasingly sub-optimal for higher SNR. Next, we study, in

more detail, the Bell Labs Layered Space-Time architecture, in particular, the original diagonal-BLAST (D-BLAST) version. We show that it has the potential to achieve channel capacity but has practical problems. We also present numerical simulation results. In section 5.4, we investigate three variations of the D-BLAST architecture that avoids some of its problems, and provide theoretical analysis using a common framework based on the multiple access channel. We demonstrate that joint decoding, if it can be accomplished, has significant advantage over successive cancellation based decoding. In the two sections that follow, we explore the possibility of combining hard and soft decision error correction coding with the tilted-QAM code proposed in section 4.3. We describe the coding scheme, present numerical simulation results, and compare them with that achieved by D-BLAST. We conclude and summarize in section 5.7.

## 5.2 OSTBC

Earlier in section 4.2, we reviewed the orthogonal space-time block codes, which was first introduced by Alamouti [1], and later extended by Tarokh [34]. We described the OSTBC as a short and smart repetition code for the two-transmit two-receive antenna systems.

In this section, we discuss how it can be concatenated with long and powerful error correction codes and what the overall system can achieve. We show that using OSTBC around a  $2 \times 2$  multiple antenna channel essentially transforms it to two independent AWGN channels. As a consequence, we can apply additional long and powerful ECC naturally. We see that the resulting capacity achieved by the overall system is near optimal in the low SNR regime.

Let us briefly summarize the OSTBC discussion in section 4.2. For a two transmit two receive antennas system, the OSTBC encodes two information symbols,  $s_1$  and

$s_2$ , into a  $2 \times 2$  transmit matrix according to

$$\mathbf{X} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}. \quad (5.1)$$

The resulting effective channel can be written as

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{12}^* \\ y_{22}^* \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{12}^* & -h_{11}^* \\ h_{22}^* & -h_{21}^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} w_{11} \\ w_{21} \\ w_{12}^* \\ w_{22}^* \end{bmatrix}. \quad (5.2)$$

The repetition, which transmits each symbol twice by both antennas and in different times, allows OSTBC to achieve the maximum diversity gain of  $N_t N_r = 4$ . However, this repetition also causes the OSTBC to lose multiplex gain. For this reason, OSTBC should only be used at low SNR, and not at high SNR.

### 5.2.1 Equivalent channel

Let us now look at how OSTBC transforms two transmit two receive antenna channels to AWGN channels so that additional error correction coding can be applied.

From the effective channel expression (5.2), we can see that the two channel vectors,  $\begin{bmatrix} h_{11} & h_{21} & h_{12}^* & h_{22}^* \end{bmatrix}^T$  and  $\begin{bmatrix} h_{12} & h_{22} & -h_{11}^* & -h_{21}^* \end{bmatrix}^T$ , are orthogonal. Because of this orthogonality, there are no interference between  $s_1$  and  $s_2$ . Thus, the OSTBC effectively transforms a  $2 \times 2$  multiple antenna channel with channel matrix  $\mathbf{H}$  to two independent AWGN channels with identical gains  $\|\mathbf{H}\|$ , one for  $s_1$  and one for  $s_2$ , as depicted in Figure 5-1. Notice that due to the repetition, only one symbol is actually transmitted in one time slot.

Once the multiple antenna channel is transformed to AWGN channels, ECC that was originally design for AWGN channel can now be applied naturally as an outer code. The concatenated system is shown in Figure 5-2. One information bit stream is error correction encoded and then demultiplexed and modulated into the symbols

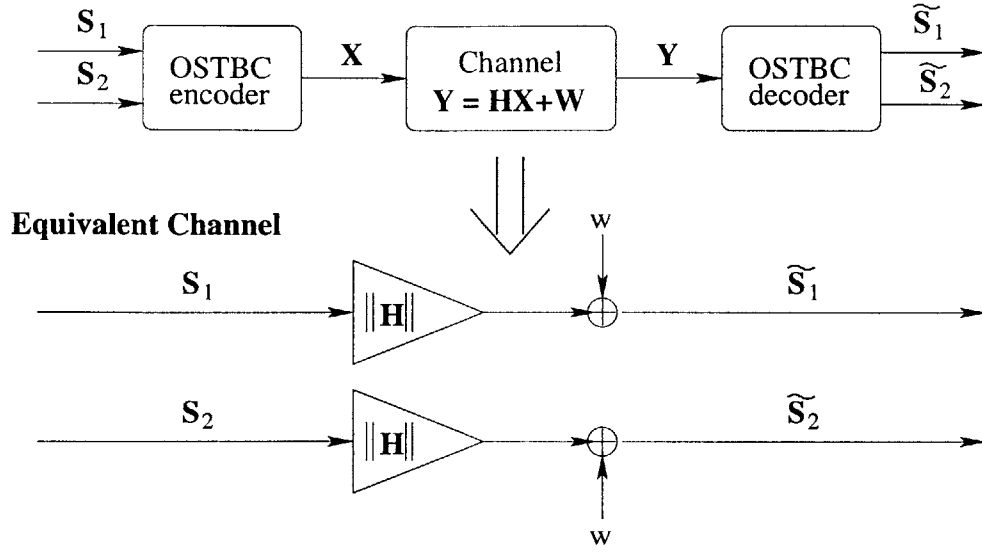


Figure 5-1: OSTBC effectively transforms a  $2 \times 2$  multiple antenna channel to two independent AWGN channels with identical gains  $\|\mathbf{H}\|$ .

streams  $S_1$  and  $S_2$ . They are then encoded by OSTBC, passed through the multiple antenna channel, and decoded by a corresponding OSTBC decoder; in other words, they each pass through the equivalent AWGN channels shown in Figure 5-1. The OSTBC decoder outputs,  $\tilde{S}_1$  and  $\tilde{S}_2$ , are then demodulated, multiplexed, and error correction decoded. Note, we can use one encoder for both  $S_1$  and  $S_2$ , or we can use separate ones. The advantage of using one encoder is that the delay is cut in half for the same code length.

### 5.2.2 Achievable Performance

Let us now look at what performance can be achieved by the concatenated system shown in Figure 5-2, and compare that with the ultimate performance achievable by any system.

If we can use capacity achieving ECC as the outer code, i.e., the delay and the complexity are affordable, then the system in Figure 5-2 should achieve the capacity

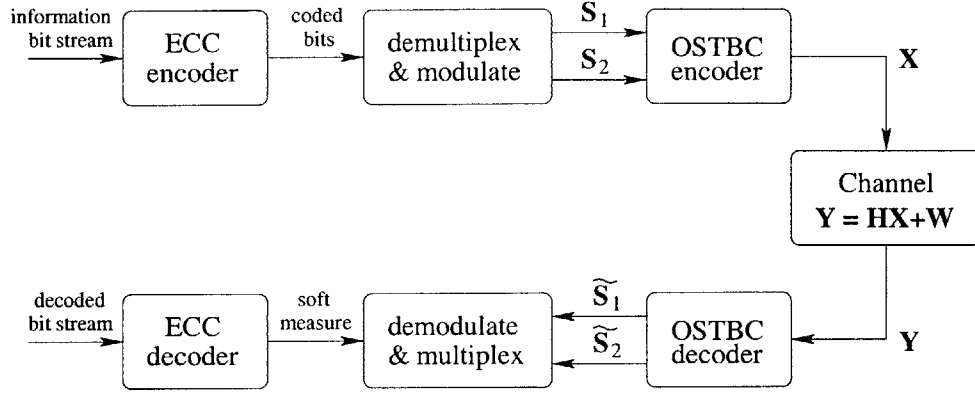


Figure 5-2: Concatenation of an OSTBC inner code with an error correction outer code.

of the equivalent channel depicted in Figure 5-1, which is,

$$C_{\text{OSTBC}}(\mathbf{H}) = \log_2 \left( 1 + \frac{\text{SNR}}{N_t} \|\mathbf{H}\|^2 \right). \quad (5.3)$$

In comparison, the channel capacity of a multiple antenna channel is

$$C_{\text{channel}}(\mathbf{H}) = \log_2 \left( \det \left( \mathbf{I}_{N_r} + \frac{\text{SNR}}{N_t} \mathbf{H} \mathbf{H}^\dagger \right) \right). \quad (5.4)$$

In the  $2 \times 2$  case, this can also be written as

$$C_{\text{channel}}(\mathbf{H}) = \log_2 \left( 1 + \frac{\text{SNR}}{N_t} \|\mathbf{H}\|^2 + \left( \frac{\text{SNR}}{N_t} \right)^2 |\det(\mathbf{H})|^2 \right). \quad (5.5)$$

Compare  $C_{\text{OSTBC}}$  (5.3) with  $C_{\text{channel}}$  (5.5), we see that the  $\left( \frac{\text{SNR}}{N_t} \right)^2 |\det(\mathbf{H})|^2$  term is missing. At high SNR, this term dominates, and its absence causes the loss of multiplexing gain. However, at low SNR, this term is insignificant. Therefore, at low SNR, we expect concatenation of an OSTBC inner code with a powerful error correction outer code to be able to effectively achieve capacity. This is true for any particular realization of  $\mathbf{H}$ . As a consequence, it is also true for an ensemble of  $\mathbf{H}$  in the case of fading channels.

To verify the above statement numerically, we plot the outage probability as



a function of SNR for various target rates  $R$  for both  $C_{\text{channel}}$  and  $C_{\text{OSTBC}}$ , i.e.,  $P[C_{\text{channel}}(\mathbf{H}, \text{SNR}) < R]$  and  $P[C_{\text{OSTBC}}(\mathbf{H}, \text{SNR}) < R]$ , in Figure 5-3.

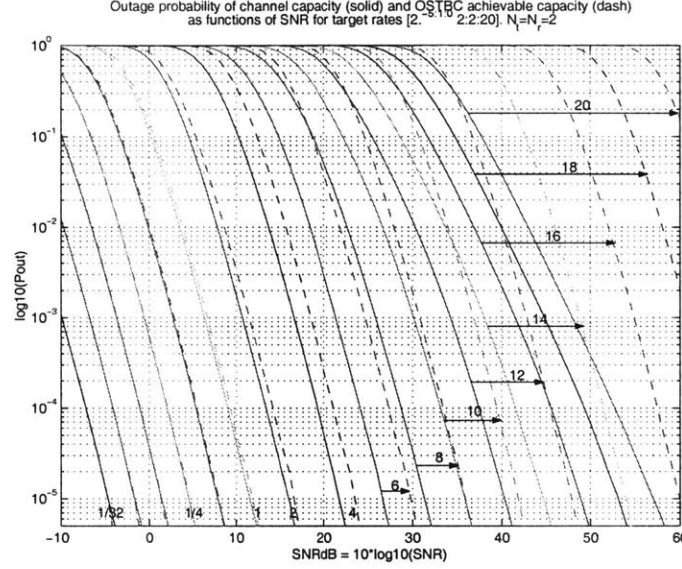


Figure 5-3: Comparison of the family of channel outage probability curves (solid) and the family of OSTBC outage probability curves (dash) as functions of SNR for rates  $2^{-5}, 2^{-4}, \dots, 1, 2, 4, 6, \dots, 18, 20$ .

From Figure 5-3, we see that for low rates, OSTBC curves match the channel outage probability curves very well as expected. The approximation is good up to about 2 b/s/Hz, at which point the gap is about 1 dB. After that, the gap starts to increase and will increase indefinitely as rate increases further, because of the difference in multiplexing gain. For rates slightly above 2 b/s/Hz, we might still want to use OSTBC for the benefits of ease of implementation and low complexity. At 6 b/s/Hz, which is quite large for current practical applications, the gap is a little over 3 dB. Beyond this point, the gap might be too large to be tolerated, where we probably would not want to use OSTBC.

In summary, OSTBC transforms multiple antenna channels to AWGN channels, and powerful capacity approaching error correction codes designed for AWGN channels can then be concatenated with it. The overall system is near optimal in the low SNR regime for rates below 2 b/s/Hz. As rates increase, the gap increases. Therefore,

for system designers who are only interested in using two antennas and transmitting at below 2 b/s/Hz, OSTBC with ECC is a highly desirable scheme. For other scenarios, other coding schemes should be considered.

In the next section, we look at a scheme that can be applied to any number of antennas, as well as a wide range of SNR levels.

## 5.3 Diagonal-BLAST

In this section, we study the Bell Labs Layered Space-Time architecture, in particular, the original diagonal-BLAST version, which is a sequential encoding and decoding method.

The D-BLAST architecture was first introduced by Foschini [9] in 1996. This scheme can be applied to systems with any number of antennas, and can be implemented with reasonably low complexity. It can also operate in a wide range of SNR and rate levels.

We first describe the diagonal layered encoding structure and two decoding algorithms, nulling and minimum mean squared estimation. We show that D-BLAST-MMSE has the potential to achieve channel capacity in the two transmit two receive antenna case. Next, we discuss several practical problems of D-BLAST, such as error propagation and some issues related to discreteness. We also run numerical simulations to see how well D-BLAST can do in practice. This result is later compared to other systems.

### 5.3.1 Layered Encoding

D-BLAST encoding done in diagonal layers is illustrated in Figure 5-4. Each row of the grid corresponds to what is transmitted by one antenna, and each column represents what is transmitted in  $\tau$  consecutive times. For example, layer “a” corresponds

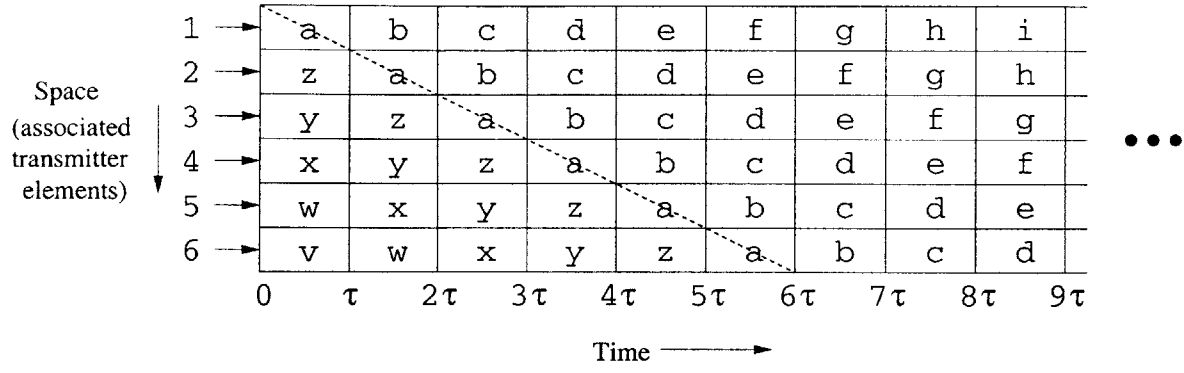


Figure 5-4: BLAST encodes in diagonal layers labeled with different alphabetical letters.

to the following entries in the transmitted signal matrix,

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,\tau} & & & \\ & & & x_{2,\tau+1} & \cdots & x_{2,2\tau} \\ & & & & \ddots & \\ & & & & & x_{N_t, N_t\tau+1-\tau} & \cdots & x_{N_t, N_t\tau} \end{bmatrix}. \quad (5.6)$$

To perform encoding, we first encode raw information bits into codewords of length  $N_t\tau$  using any suitable coding scheme. Each codeword is then associated with one diagonal layer (with  $N_t\tau$  entries) for transmission during the appropriate slots according to Figure 5-4.

The key features of this encoding scheme are

- 1) all  $N_t\tau$  symbols of a codeword are transmitted during different times, and
- 2) each codeword is transmitted (in pieces) by all antennas.

The reason for transmitting one codeword using all possible antennas is to maximize the tolerance of some of the channel coefficients being in deep-fade. The reason for transmitting in different times is so that the symbols from the same codeword do not interfere with each other, which allows for convenient decoding as we illustrate next.

### 5.3.2 Layered Decoding

In D-BLAST decoding, the diagonally-layered codewords are decoded one at a time, in order, via successive cancellation. We first briefly describe how successive cancellation is done to handle the interference between the layers, then describe how each layer can be decoded.

Suppose we want to decode the layer labeled “a” in Figure 5-4. By this time, layers “z” and before should have already been decoded. Therefore, we can completely cancel out their interference on layer “a”. However, layers “b” and later have not been decoded, so their interference remains. Two ways of handling these interference are described in detail in the next two sections.

After handling the interference between the layers, each symbol of layer “a” is then simply corrupted by some effective additive noise. There might be different amount of noise on different symbols, and the coding applied would allow symbols that are more reliable to help decode the ones that are not.

We decode each layer as if it had just gone through a varying gain AWGN channel. One way is to do a two stage decoding. First all the symbols in that layer are individually detected. The intermediate result can be in the form of either soft decision or hard decision. The entire block is then passed on to the decoder where the original information is extracted.

Next, let us describe in more detail the two ways of handling the interference between layers, BLAST-nulling and BLAST-MMSE, and see what performance can be achieved. These two methods are both based on successive cancellation and differ in the way they handle the interference from the layers that have not been decoded.

#### **BLAST-nulling**

The BLAST-nulling scheme was earlier reviewed in section 3.2, and is briefly summarized here. BLAST-nulling uses successive cancellation to cancel out interference from layers already decoded and use Gram-Schmidt or QR factorization to null out layers that have not been by only looking in the dimension orthogonal to all the

interference.

Let us suppose that we want to detect the entry  $x_{2,2\tau}$  of layer “a”. The received signal vector at time  $2\tau$  is  $\mathbf{y}_{2\tau} = \mathbf{H}\mathbf{x}_{2\tau} + \mathbf{w}_{2\tau}$ . The entries  $x_{i,2\tau}$ ,  $i > 2$ , have been decoded, and  $x_{1,2\tau}$  has not been.

We first factorize the channel matrix as  $\mathbf{H} = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q}$  is unitary and  $\mathbf{R}$  is upper triangular.  $\mathbf{y}_{2\tau}$  can then be pre-processed to obtain  $\mathbf{y}'_{2\tau} = \mathbf{Q}^\dagger \mathbf{y}_{2\tau} = \mathbf{R}\mathbf{x}_{2\tau} + \mathbf{w}'_{2\tau}$ , where  $\mathbf{w}'_{2\tau} = \mathbf{Q}^\dagger \mathbf{w}_{2\tau}$  and  $\dagger$  denotes the conjugate transpose operation, so

$$\begin{bmatrix} y'_{1,2\tau} \\ y'_{2,\tau} \\ \vdots \\ y'_{N_t,2\tau} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N_t} \\ 0 & r_{22} & \cdots & r_{2N_t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{N_t N_t} \end{bmatrix} \begin{bmatrix} x_{1,2\tau} \\ x_{2,2\tau} \\ \vdots \\ x_{N_t,2\tau} \end{bmatrix} + \begin{bmatrix} w'_{1,2\tau} \\ w'_{2,2\tau} \\ \vdots \\ w'_{N_t,2\tau} \end{bmatrix}. \quad (5.7)$$

Focus on the second row of the above matrix equation,

$$y'_{2,2\tau} = r_{22}x_{2,2\tau} + r_{23}x_{3,2\tau} + \cdots + r_{2N_t}x_{N_t,2\tau} + w'_{2,2\tau}, \quad (5.8)$$

we see that the undecoded entry  $x_{1,2\tau}$  does not appear due to the nulling, and the already decoded entries  $x_{i,2\tau}$ ,  $i > 2$ , can be canceled out, leaving

$$\tilde{y}_{2,2\tau} = r_{22}x_{2,2\tau} + w'_{2,2\tau}. \quad (5.9)$$

Now, we can detect  $x_{2,2\tau}$ . Notice that if some of the entries  $x_{i,2\tau}$ ,  $i > 2$  were mis-decoded, wrong values would have been canceled out, and  $x_{2,2\tau}$  might be mis-decoded as well. This phenomenon is known as error propagation.

Once we detect all entires of layer “a” in similar fashion, we send the entire block into a decoder to correct for any detection errors and get the transmitted codeword. The BLAST-nulling decoding scheme is summarized in Figure 5-5.

To see what fraction of the total capacity can be achieved by such a system, we note that the effective channel gain experienced by  $x_{2,2\tau}$  is  $r_{22}$ . Similarly, the channel gain experienced by entry  $x_{i,j}$  is  $r_{ii}$ . So by using the BLAST architecture

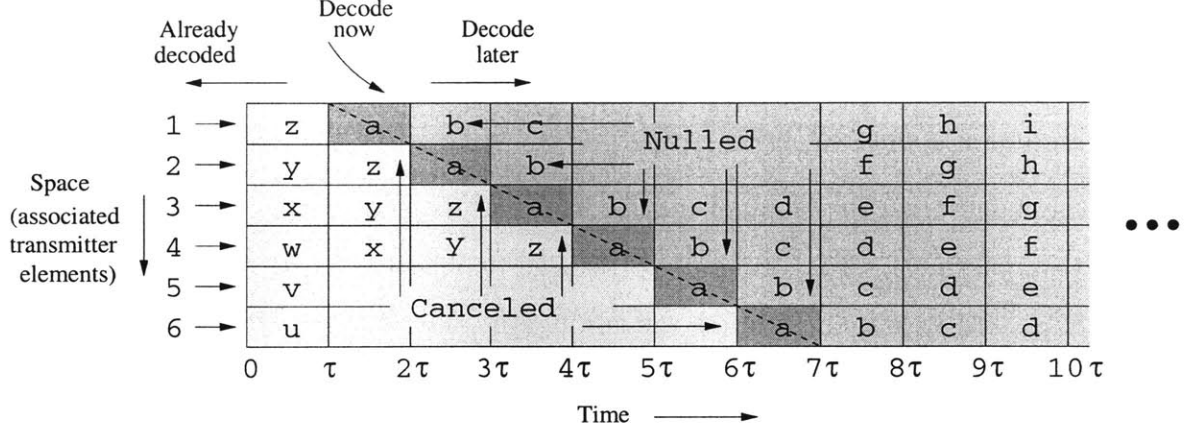


Figure 5-5: BLAST-nulling decoding scheme. Interference from symbols in later layers, which are not yet decoded, are nulled out via QR factorization of the channel matrix. Interference from symbols in previous layers, which are already decoded, are eliminated using successive cancellation.

with BLAST-nulling decoder, the multiple antenna channel is transformed into  $N_t$  independent AWGN channels, each with gain  $r_{ii}$ . This is similar to the transformation in the OSTBC case, shown in Figure 5-2, except that the channels gains are different and each codeword goes through all these different sub-channels due to the diagonal structure.

The channel capacity of each effective scalar sub-channel is  $\log_2(1 + \rho r_{ii}^2)$ , where  $\rho = \text{SNR}/N_t$ . Therefore, the total capacity achievable by BLAST-nulling is

$$C_{\text{BLAST-nulling}} = \sum_{i=1}^{N_t} \log_2(1 + \rho r_{ii}^2). \quad (5.10)$$

Compare to the ultimate capacity of the Gaussian channel,

$$C_{\text{channel}}(\mathbf{H}) = \log_2(\det(I_{N_r} + \rho \mathbf{H} \mathbf{H}^\dagger)) = \log_2(\det(I_{N_r} + \rho \mathbf{R} \mathbf{R}^\dagger)),$$

BLAST-nulling is sub-optimal. For example, for  $N_t = 2$ ,

$$C_{\text{BLAST-nulling}} = \log_2((1 + \rho r_{11}^2)(1 + \rho r_{22}^2)), \quad (5.11)$$

$$C_{\text{channel}} = \log_2((1 + \rho r_{11}^2)(1 + \rho r_{22}^2) + \rho r_{12}^2). \quad (5.12)$$

BLAST-nulling utilizes only the diagonal elements of  $\mathbf{R}$  and disregards the off diagonal terms, causing it to be sub-optimal. In the limit of high SNR, the difference can become arbitrarily small. However, it is still less robust against fading, since having small diagonal terms for  $\mathbf{R}$  would be sufficient to kill the transmission. We do not get any protection from the off diagonal terms. Although BLAST-nulling is sub-optimal, it is an efficient decoding scheme.

## BLAST-MMSE

BLAST-MMSE is a variation of BLAST-nulling. In this section, we describe how they differ and analyze the performance achievable by BLAST-MMSE in the two-transmit two-receive antenna case. We show that while BLAST-nulling is sub-optimal, BLAST-MMSE can actually achieve the full channel capacity by utilizing the off diagonal terms of  $\mathbf{R}$ .

The main difference between BLAST-MMSE and BLAST-nulling is how they handle interference from entries that have not been decoded. In equation (5.7), where  $\mathbf{y}'_{2\tau} = \mathbf{R}\mathbf{x}_{2\tau} + \mathbf{w}'_{2\tau}$  is written out in full matrix form, instead of focusing only on the second row to decode  $x_{2,2\tau}$ , we use the first two rows. We subtract out the already decoded terms  $x_{i,2\tau}$ ,  $i > 2$ , leaving

$$\tilde{\mathbf{y}}_{2,2\tau} = \begin{bmatrix} \tilde{y}_{1,2\tau} \\ \tilde{y}_{2,2\tau} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} x_{1,2\tau} \\ x_{2,2\tau} \end{bmatrix} + \begin{bmatrix} w'_{1,2\tau} \\ w'_{2,2\tau} \end{bmatrix}. \quad (5.13)$$

Since we want to detect  $x_{2,2\tau}$  only, we can treat  $x_{1,2\tau}$  as noise, combine the two noise terms, and rewrite the above equation as

$$\tilde{\mathbf{y}}_{2,2\tau} = \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix} x_{2,2\tau} + \begin{bmatrix} r_{11} \\ 0 \end{bmatrix} x_{1,2\tau} + \begin{bmatrix} w'_{1,2\tau} \\ w'_{2,2\tau} \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix} x_{2,2\tau} + \begin{bmatrix} v_{1,2\tau} \\ v_{2,2\tau} \end{bmatrix}. \quad (5.14)$$

We can now find the MMSE of  $x_{2,2\tau}$  using the appropriate noise covariance matrix. It turns out that the resulting effective SNR is  $\rho r_{22}^2 + \rho r_{12}^2 / (1 + \rho r_{11}^2)$ , instead of  $\rho r_{22}^2$  as is the case in BLAST-nulling. So by using the BLAST architecture with

BLAST-MMSE decoder, the  $2 \times 2$  multiple antenna channel is transformed into two independent AWGN channels with gains  $r_{11}$  and  $\sqrt{r_{22}^2 + r_{12}^2 / (1 + \rho r_{11}^2)}$ . as shown in Figure 5-6.

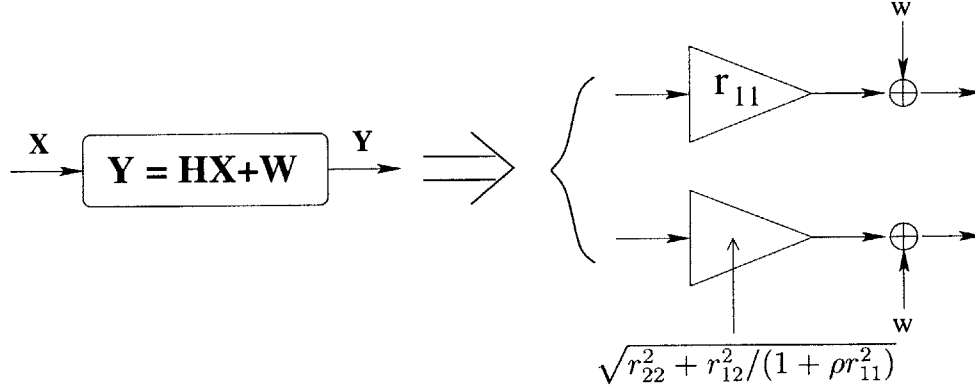


Figure 5-6: BLAST-MMSE effectively transforms a  $2 \times 2$  multiple antenna channel to two independent AWGN channels with effective gains  $r_{11}$  and  $\sqrt{r_{22}^2 + r_{12}^2 / (1 + \rho r_{11}^2)}$ .

The total capacity achieved by BLAST-MMSE is

$$C_{\text{BLAST-MMSE}} = \log_2 (1 + \rho r_{11}^2) + \log_2 \left( 1 + \rho r_{22}^2 + \frac{\rho r_{12}^2}{1 + \rho r_{11}^2} \right) \quad (5.15)$$

$$= \log_2 ((1 + \rho r_{11}^2)(1 + \rho r_{22}^2) + \rho r_{12}^2) = C_{\text{channel}} \quad (5.16)$$

This indicates that BLAST-MMSE achieves full channel capacity.

Although theoretically BLAST-MMSE is optimal and BLAST-nulling is near optimal, they do have several practical problems, which we discuss in the next section.

### 5.3.3 D-BLAST Caveats

In this section, we discuss some of the practical issues associated with the D-BLAST architecture. Some of them are associated with the diagonal layered nature of D-BLAST; another is associated with the discrete nature of the constellations that are often used in practice.



## Layered Structure Problems

Some of the problems with the diagonal layered structure are error propagation, additional re-initialization cost, and increased delay.

Let us first discuss *error propagation*. Because decoding later layers requires the previous layers to be correctly decoded, once one layer is mis-decoded, the error will propagate on to later layers and may not stop for several layers. To reduce error propagation, we must protect each layer with sufficiently strong error correction codes. However, even if we do so, there might still be unpredictable events which could cause occasional errors. If each of such events causes subsequent errors due to propagation, it would make the system less robust.

One sure way to stop error propagation is to reinitialize, stop transmitting for several layers and start transmitting a new layer without having to cancel previous ones. However, this would increase the overhead associated with the initialization. Because of the diagonal layered architecture, the lower triangle before the first layer must all be initialized to zero or some known value to allow decoding of the first layer. In the case of  $N_t = 2$ , this overhead is equal to half of a code block. For larger  $N_t$ , the overhead would be even greater. To reduce the impact of overhead, we would want to transmit many layers before re-initializing. However, this would lead us back to the error propagation problem. Also, if the channel varies sufficiently fast and come in and out of fade every few layers, then due to the error propagation, we must reinitialize every time the channel comes out of a fade. This would lead to a lot of re-initialization overhead.

The last problem associated with the layered structure that we would like to mention is increased delay. For a given code length, spreading the codeword out in a diagonal form so that only one symbol is transmitted at a time increases the delay by a factor of  $N_t$ , compared to simply using all antennas to transmit at the same time. Accompanying the increased delay, there is also increased buffering need, which for long codewords (needed to reduce error propagation) might cause a practical problem.

## Discrete Interference

One problem associated with BLAST-MMSE is that it treats all interference from layers that have not been decoded as Gaussian noise, while in practice, they are often chosen out of QAM-like constellations. This mis-match could lead to performance degradation that are unnecessary.

More specifically, from (5.14), we see that when  $x_2$  (let us drop the timing index for convenience) is being decoded, the effective noise is a combination of  $x_1$  and the actual additive white Gaussian noise  $w'$ . If  $x_1$  has Gaussian distribution, the combined noise would also be Gaussian. In this case, the MMSE estimator is also the ML estimator, making MMSE a very good choice. However, in practice, Gaussian input distribution is never used. Instead, regular constellation such as 16-QAM and 64-QAM are used. In this case, the combined noise would not be Gaussian. In fact, if  $r_{11}$  is really large, the noise distribution would look like a set of impulses, very different from Gaussian. They simply have the same variance. It is well known that Gaussian distribution has the largest entropy for a given variance. Therefore, the Gaussian noise approximation would be overly pessimistic.

We might be able to take advantage of the fact that  $x_1$  is discrete by treating the signal constellation as a lattice rather than incorrectly treating it as a continuous Gaussian distribution. For example, when the channel is near singular, i.e.,  $r_{11}$  is much larger compared to  $r_{22}$ , the received constellation points  $\mathbf{H}\mathbf{x}$  might be close to being “co-planer”, but they might still be well separated. We would be able to tell which constellation point is transmitted. However, if we treat  $x_1$  as a continuous Gaussian noise with a large variance, we would have a hard time detecting  $x_2$ . Later in section 5.6, we use a lattice-aware detector to treat the discreteness of the constellation. The simulation result shows that there is in fact a small gain.

## Finite Constellation Size Problem

Another practical problem associated with BLAST is what we call the finite constellation size problem. The problem is that the amount of information that can be

carried is not only limited by the channel capacity, but also by the constellation size used. Recall that BLAST effectively transforms a multiple antenna channel to multiple single antenna channels. The stronger sub-channels are expected to carry more information than the weaker sub-channels. However, if the constellation used is too small, then stronger sub-channels might not be able to carry as much information it otherwise can. This could potentially prevent the total channel capacity from being achieved.

In this section, we investigate how this phenomenon affects the system performance. We show using both theoretical analysis and numerical simulation that using a constellations that is too small can lead to a loss of diversity gain. We then discuss how to choose the constellation size so that the performance loss is acceptable and the constellation is not unnecessarily large. We propose to set the constellation size to  $M^2 = \min(1 + \text{SNR}, 2^R)$  or slightly larger.

In the two transmit two receive antenna case, the BLAST effectively transforms the multiple antenna channel to two sub-channels. When MMSE detection is used, combined capacity achievable is (5.15)

$$C_{\text{BLAST-MMSE}} = \log_2(1 + \rho r_{11}^2) + \log_2\left(1 + \rho r_{22}^2 + \frac{\rho r_{12}^2}{1 + \rho r_{11}^2}\right).$$

Achieving this capacity assumes usage of Gaussian input distribution. However, in practice, only finite constellations are used, in which case, the capacity achievable by each sub-channel is upper bounded by  $\log_2(M^2)$ , where  $M^2$  is the size of the QAM constellation used. So the total capacity achievable is instead

$$C_{\text{BLAST-MMSE,M}} = \log_2(\min(1 + \rho r_{11}^2, M^2)) + \log_2\left(\min\left(1 + \rho r_{22}^2 + \frac{\rho r_{12}^2}{1 + \rho r_{11}^2}, M^2\right)\right).$$

Ideally, we would like to have  $C_{\text{BLAST-MMSE,M}} > R$  whenever  $C_{\text{BLAST-MMSE}} > R$ , so that there is no loss in outage probability due to finite constellation size.

The problem associated with finite constellation arises when the constellation size is small compared to the rate, i.e.,  $M^2 < 2^R$ . In this case, if one of the sub-channel is in a sufficiently deep fade, i.e., the associated capacity is less than  $R - \log_2(M^2)$ ,

then, no matter how large the other channel gain is, the overall capacity would be less than  $R$ , which would cause the transmission to fail. Therefore, both sub-channels must have sufficient gain to support a minimum rate of  $R - \log_2(M^2)$  in order for the transmission to succeed. For example, suppose we want to transmit at 8 b/s/Hz and choose to use 64-QAM constellation (and rate 2/3 code), each sub-channel can carry at most 6 b/s/Hz. If any of the sub-channels supports less than 2 b/s/Hz, the transmission would fail. As we see, using a constellation that is too small makes the system less robust; fading of any sub-channels would cause the transmission to fail. This translates to a loss of diversity gain.

To demonstrate the loss of diversity gain numerically, we compute the probability of  $C_{\text{BLAST-MMSE},M} < R$  for  $R = 8$  b/s/Hz and  $M^2 = 16, 64$ . We then compare them to the channel outage probability,  $P[C_{\text{channel}} = C_{\text{BLAST-MMSE}} < R]$ , as shown in Figure 5-7. The bottom curve is the channel outage probability, which corresponds to  $M^2 \geq 2^R = 256$ . It has slope approaching 4. For  $M^2 = 64$  (middle curve) and  $M^2 = 16$  (top curve), the limiting slopes are only 1, with the  $M^2 = 16$  case performing slightly worse by a constant factor.

The analytical justification for the diversity reduction to 1 is that with constellation size  $M^2 < 2^R$ , the dominating outage event is when the capacity associated with the second sub-channel,  $\log_2 \left( 1 + \rho r_{22}^2 + \frac{\rho r_{12}^2}{1 + \rho r_{11}^2} \right)$ , is less than  $R - \log_2(M^2)$ , which is a positive constant. This event typically happens when  $\rho r_{22}^2$  is small and  $r_{12}^2 \approx r_{11}^2$ , which happens with probability on the order of  $\rho^{-1}$ .

Next, we would like to explore how the constellation size should be chosen so that the performance loss is reasonably small. One obvious solution is to always set  $M^2 = 2^R$ , i.e., each sub-channel is capable of supporting the entire rate on its own even when all the other sub-channels completely fade. The problem with this solution is that the constellation size required might be too large.

Another solution is inspired by the realization that smaller constellations might be sufficient at relatively low SNR. In Figure 5-7, we see that the middle curve is quite close to being optimal below 18 dB. We propose to set the constellation size to  $M^2 = \min(1 + \text{SNR}, 2^R)$ . This would allow us to use smaller constellation sizes at

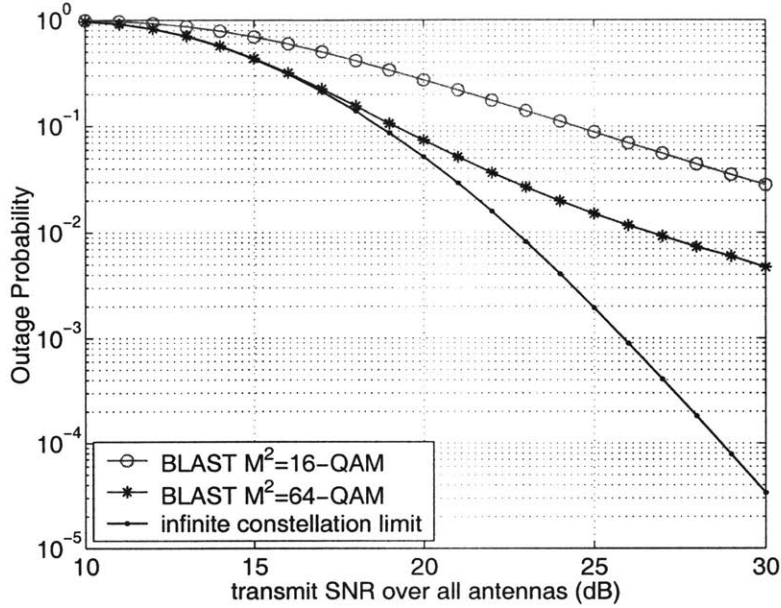


Figure 5-7: Demonstration of the finite constellation size problem. When the constellation size used is too small, there is a loss of diversity gain.

lower SNR. At high SNR, we still set  $M^2 = 2^R$  to avoid loss of diversity.

To demonstrate how well the above proposal works numerically, we plot outage probability curves for various rates as functions of SNR in Figure 5-8. The set of solid curves correspond to channel capacity outage curves and the dashed one correspond to  $P[C_{\text{BLAST-MMSE},M} < R]$ , where  $M^2 = \min(1 + \text{SNR}, 2^R)$ . We see that the two sets of curves match very well. The performance loss is quite small as we desired.

In summary, when using the BLAST architecture, which transforms the multiple antenna channel to multiple single antenna channels, we need to make sure that the constellation used is sufficiently large. Otherwise, there might be a loss of diversity gain. We show that using  $M^2 = \min(1 + \text{SNR}, 2^R)$  for the two transmit two receive antenna case results in reasonably small performance loss. However, the constellation size required might still be quite large at times.

While BLAST architecture requires usage of large constellations, we do not believe this is intrinsic to all coding schemes. With two antennas, the total constellation size is  $(M^2)^2 = M^4$ . With the right coding scheme, we should only need  $M^2 = 2^{R/2}$  to

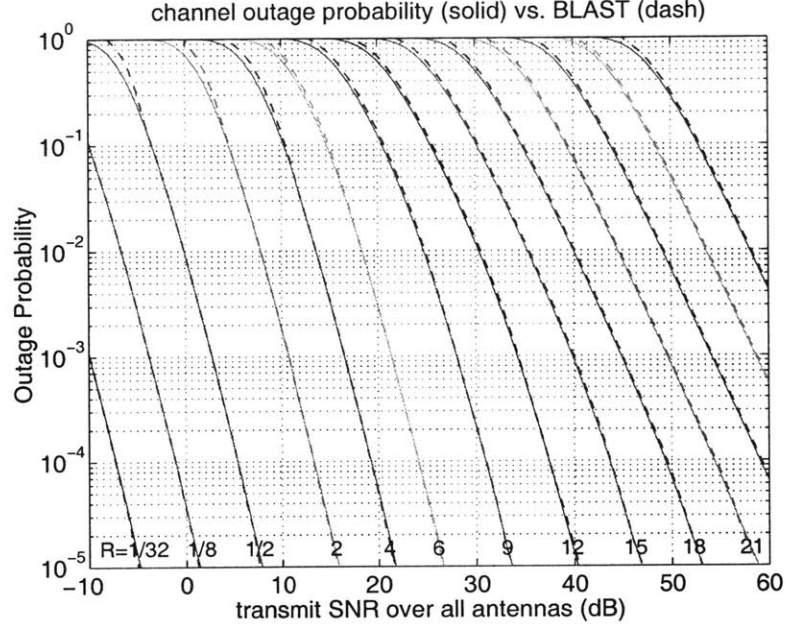


Figure 5-8: If we select the constellation size to be  $M = \log_2(1+\text{SNR})$ , then the outage probability associated with BLAST (dash) seems to be very close to the ultimate channel outage probability (solid). The loss due to finite constellation effect is small.

support rate  $R$ , much less than  $M^2 = 2^R$ . Later in section 5.6, we will see that it is indeed possible to use smaller constellations and achieve similar performance.

### 5.3.4 Experimental Setup

We perform some numerical simulations to see how well the D-BLAST-MMSE scheme works in practice. The experimental setup is describe in detail in this section and the simulation results are presented in the next section. They will be compared to other coding schemes we evaluate later.

For simplicity, let us ignore the various issues associated with the layered nature of D-BLAST mentioned earlier in section 5.3.3. We would simply keep these issues in mind when we make our comparisons later. To be more specific, we simulate the error rate for the first layer only, so that there is no error propagation. We do not count the initialization overhead in our rate calculation. We also grant the additional delay and buffering need by BLAST for free and simply use code-length as the length

measure.

Let us now describe the experimental setup in detail. First, the encoding process is done in four steps.

1. A block of binary information bits is first encoded into a block of coded bits using a powerful error correction code, in particular, a length 1024 (information bits) low density parity check (LDPC) code <sup>1</sup>.
2. The coded bits are randomly interleaved so that bits nearby go through different sub-channels, and are not modulated into the same symbol, which could cause correlated errors.
3. The bits are modulated into complex symbols with real and imaginary parts taking values such as  $\{\pm 1, \pm 3, \pm 5, \pm 7, \dots\}$ . We use regular constellations, such as 64-QAM and 256-QAM. We choose to use Gray-labeling as shown in Figure 5-9. This way, confusion between neighboring symbols would lead to only one bit error. For example, there is only one bit difference between  $-7 \sim (0, 1, 1)$  and  $-5 \sim (0, 1, 0)$ . Gray-labeling is also used in systems like bit interleaved coded modulation (BICM) [4].

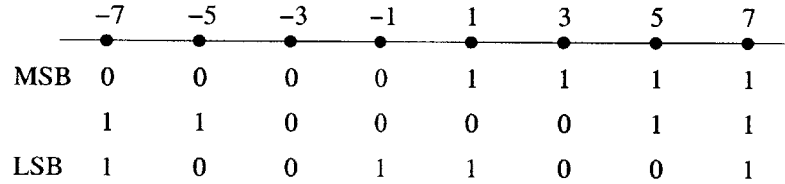


Figure 5-9: Gray-labeling with 8-PAM constellation.

4. The symbols are arranged into a matrix in the following manner.

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,\tau} & \text{some} & \text{random} & \text{symbols} \\ 0 & \cdots & 0 & x_{2,\tau+1} & \cdots & x_{2,2\tau} \end{bmatrix}$$

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<sup>1</sup>Software for Low Density Parity Check Codes was developed by Radford M. Neal, Dept. of Statistics and Dept. of Computer Science, University of Toronto.

Half of the symbols are transmitted by antenna one for half a block, while antenna two is off. The second half of the symbols are transmitted by antenna two for the second half of the time while antenna one transmits some random symbols representing data from the next layer.

After encoding, the matrix  $\mathbf{X}$  is transmitted over the multiple antenna channel  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , where one  $\mathbf{H}$  matrix with IID  $\mathcal{CN}(0, 1)$  entries is generated for each block, and an independent one is used for the next one.

Decoding is also performed in four steps corresponding to the reverse of the encoding steps.

First, the multiple antenna channel is transformed into two equivalent scalar sub-channels using the layered decoding algorithm described in section 5.3.2. If we choose to use BLAST-MMSE, then the two equivalent channels would have SNR  $\rho r_{11}^2$  and  $\rho r_{22}^2 + (\rho r_{12}^2)/(1 + \rho r_{11}^2)$ . Note, for the second sub-channel, the interference (from  $x_1$ ) is discrete, but are treated as Gaussian noise.

With each equivalent scalar channel taking the form of  $y = x + w$  with certain SNR, we can now use decoding techniques for the AWGN channel. We first compute the bit-wise log likelihood ratios (LLR), i.e.,  $\log(P[y|b_i = 1]/P[y|b_i = 0])$ , for each bit  $b_i$  used to label  $x$ . When the constellation is binary, we have

$$\log \left( \frac{P[y|b = 1]}{P[y|b = 0]} \right) = \log \left( \frac{P[y|x = +1]}{P[y|x = -1]} \right) = \log \left( \frac{e^{-(y-1)^2/2\sigma_w^2}}{e^{-(y+1)^2/2\sigma_w^2}} \right) = \frac{(y+1)^2 - (y-1)^2}{2\sigma_w^2}.$$

In this case, the LLR is just the difference between the square distances normalized by  $\sigma_w^2$ .

However, when non-binary constellations are used, there are multiple constellation points with a particular bit being 1 (or 0) as shown in the Gray-labeling picture in Figure 5-9. In this case, we approximate the LLR by only considering the contribution from the closest constellation points. More specifically, for a given  $y$ , to obtain the LLR for bit  $b_i$ , we measure its distance to the closest constellation point with  $b_i = 1$  and the closest point with  $b_i = 0$ , then compute the difference. As an example, in Figure 5-10, we plot the LLR for the 3 different bits as functions of  $y$ , for the



case of  $\sigma_w^2 = 1$ , and 8-PAM constellation. This way of computing LLR is only an approximation. However, exact computation requires summing over too many constellation points, exponentially many in rate.

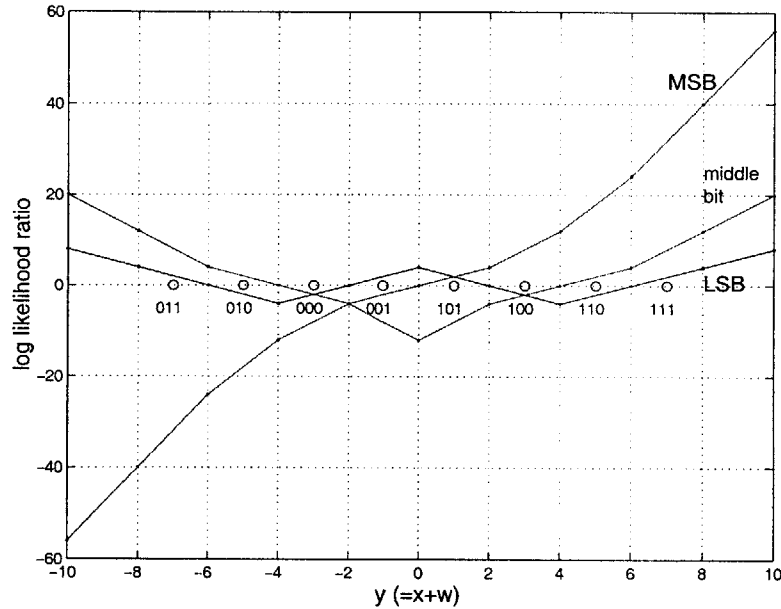


Figure 5-10: Approximations of log likelihood ratios of different bits as functions of  $y = x + w$  for an 8-PAM constellation with  $\sigma_w^2 = 1$ .

After we obtain the LLR, we then undo the interleaving done at the encoder and pass the result on to a LDPC decoder to finish decoding.

### 5.3.5 Simulation Results

Using the encoding and decoding procedures described in the last section, we perform four sets of simulations at two different rates, each using two different constellation sizes. More specifically, we evaluate 1020-bit block error rates for rate 6 b/s/Hz using 64-QAM and 256-QAM constellations and 1024-bit block error rate for rate 8 b/s/Hz using 256-QAM and 1024-QAM constellations.

The reason for working at rates 6 and 8 b/s/Hz is that they are high enough such that OSTBC is quite far from optimal while low enough such that they are of practical interest. The reason for using two different constellation sizes is to demonstrate the

finite constellation size problem discussed in section 5.3.3.

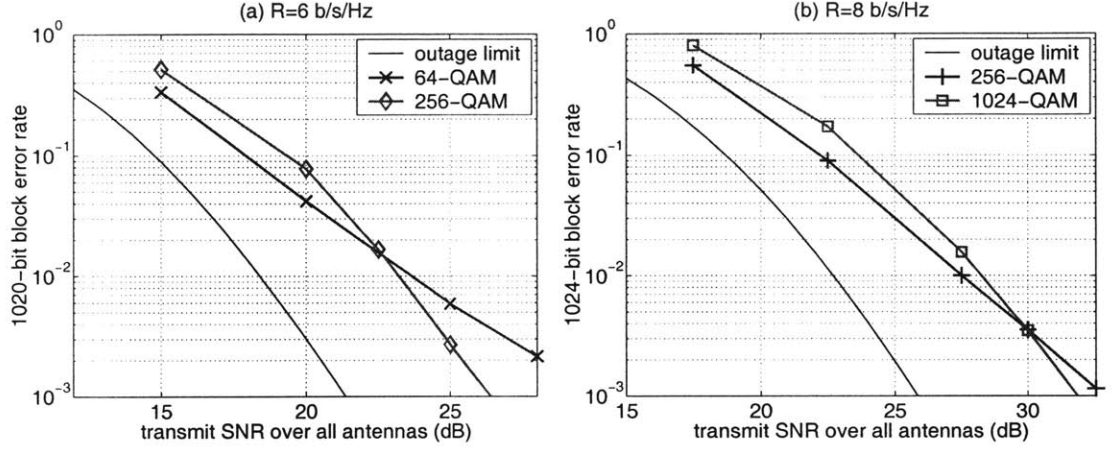


Figure 5-11: Block error rate for  $R = 6$  and  $R = 8$  b/s/Hz using D-BLAST-MMSE architecture on a two-transmit two-receive antenna system.

The simulation results are shown in Figure 5-11 in the form of block error rates vs. SNR. The left figure is for  $R = 6$  b/s/Hz, and the right one is for  $R = 8$  b/s/Hz. In each figure, the left most thin line represent the ultimate performance limit associated with channel outage probabilities, while the thicker lines are the resulting block error rates using the different constellation sizes. We see that similar trends are exhibited at both rates.

We can use the gap between the block error rates achieved and the ultimate performance limit as a measure of goodness. We see that at  $10^{-2}$  block error rate, D-BLAST-MMSE reaches 4.8 dB from capacity at both rates; while at  $10^{-3}$  block error rate, the gap is 5 dB for  $R = 6$  b/s/Hz and 6 dB for  $R = 8$  b/s/Hz.

Let us now compare the block error rates achieved by the different constellation sizes. Due to the similarity at the two rates, let us comment on only the  $R = 6$  b/s/Hz case. For this case, 64-QAM constellation is just large enough for each sub-channel to carry all 6 bits alone.

Comparing the 64-QAM and 256-QAM performance, we see that at 64-QAM performs slightly better at low SNR while 256-QAM catches up at high SNR. The reason for larger constellation to perform worst at low SNR is that the points become

too close compared to the noise level. When this happens, noise could carry the original signal to many constellation points away. This makes decoding difficult. In particular, our log likelihood ratio approximation, which only considers the closest neighbors, becomes inaccurate.

The reason that 256-QAM catches up is related to the finite constellation size problem. Although the 64-QAM constellation can carry 6 bits, it is insufficient because the LDPC code used is not capacity achieving. When one of the sub-channel is in sufficiently deep fade, we can not recover the original codeword just from the 6 bits carried by the remaining sub-channel. From Figure 5-11 we see that the diversity achieved by the 64-QAM constellation is smaller. This reflects the finite constellation size effect. On the other hand, using 256-QAM would provide the additional margin needed by the imperfect LDPC code. The slope achieved is similar to that of the outage probability.

In summary, we studied the D-BLAST architecture in detail. In particular, we show that D-BLAST-MMSE is theoretically optimal but have practical problems. Our numerical simulations show that D-BLAST-MMSE can reach within 5 dB from capacity.

## 5.4 Modified BLAST in Block Form

In the last section, we studied the original version of the BLAST architecture, D-BLAST, which has a sequential form. In this section, we study three variations of BLAST that are strictly block codes. The goal is to avoid the caveats associated with the sequential nature of D-BLAST and explore different coding structures.

We first introduce these three different block-form variations, V-BLAST, two-layer-D-BLAST, and X-BLAST, and describe their coding structures in section 5.4.1. We then describe one common framework based on multiple access channel in which all three schemes can be studied in section 5.4.2. The idea is to treat the multiple codewords as messages from different users. In sections 5.4.3 to 5.4.5, we investigate each scheme individually. We evaluate the achievable performance analytically, in

terms of both capacity and diversity-multiplexing tradeoff, when successive cancellation based decoding is performed and when optimal joint decoding is done. We also plot outage probability curves and compare them with the ultimate channel outage performance limit. In section 5.4.6, we compare all three schemes along with OSTBC, to see which method is the best in different SNR regimes, and where future potentials lay.

All three schemes are described and analyzed in the two transmit antenna case. The coding structure can be extended to higher dimensional cases, and the analysis can potentially be done using the same framework, maybe with somewhat increased complexity.

### 5.4.1 Code Designs

In this section, we introduce V-BLAST, two-layer-D-BLAST, and X-BLAST. The first one is well known, while the latter two are new interesting structures that have not been previously studied to our knowledge. The coding structures are described in this section, while the analysis will be postponed until after we describe the common framework in which all three schemes can be studied.

#### V-BLAST

The first variation of D-BLAST is the well-known V-BLAST, or vertical-BLAST [39], which limits coding to each row of the transmitted signal matrix  $\mathbf{X}$  as shown in Figure 5-12 <sup>2</sup>. One codeword goes in the lighter region and the other one goes in the darker region. V-BLAST was introduced after D-BLAST by the same group of people as a simplified version of the original. However, there is a sacrifice in performance. In V-BLAST, each codeword appears in only one row, i.e., coding takes place only across time, but not across space. Because of this, V-BLAST is sub-optimal in the sense that it achieves the maximum multiplexing gain, but not the maximum diversity gain, as we will see later.

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<sup>2</sup>The word vertical is used instead of horizontal because the transmitted signal matrix  $\mathbf{X}$  was written in a transposed form in the original literature, relative to our formulation.

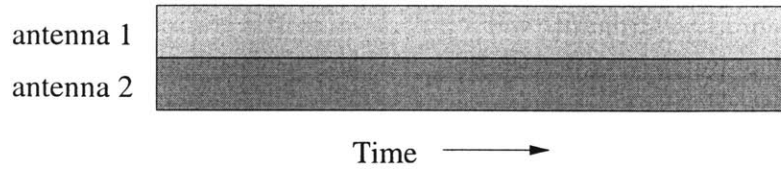


Figure 5-12: V-BLAST, where coding is restricted to one row of the transmitted signal matrix.

### Two-Layer-D-BLAST

Another variation of D-BLAST we would like to consider is to simply limit D-BLAST to two layers, as shown in Figure 5-13. In this way, both layers are the end layers. We can choose to decode each layer without decoding the other one. There would be no error propagation issue. Also, both codewords have some symbols that are always interference free. This could lead to more robust performance. However, the initialization problem is still there, i.e., nothing is being transmitted in the white region labeled with “0” in Figure 5-13. And because we reinitialize every two layers, the overhead takes a fixed and significant percentage of the total resources available. We will show later that this two layer version of D-BLAST can achieve the maximum diversity gain, but not the maximum multiplexing gain.

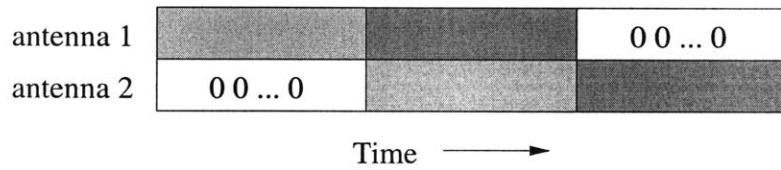


Figure 5-13: Two layers of BLAST. Both layers are end layers.

### X-BLAST

The last variation of D-BLAST we would like to study is a blend between D-BLAST and OSTBC or tilted-QAM as shown in Figure 5-14, which we refer to as *X-BLAST*. In this version, there are no overhead issue, unlike two-layer-D-BLAST, and coding occurs in both space and time, unlike V-BLAST. However, we can foresee that decoding

for this design could be difficult, since both codewords could suffer severe interference from the other one. We will see that if we can manage joint decoding, this design can achieve the channel outage probability and the optimal diversity-multiplexing tradeoff.

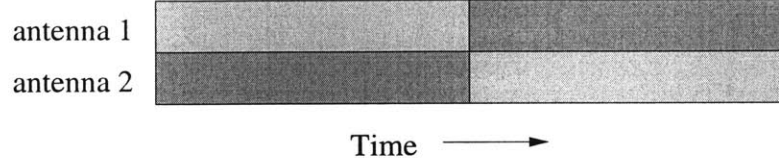


Figure 5-14: X-BLAST, where two codewords cross like in OSTBC or tilted-QAM.

### 5.4.2 Multiple Access Channel Framework

In this section, we describe one common framework based on multiple access channel, in which we can evaluate the performance achievable by the various designs proposed in the previous section. We want to analytically express the capacity achievable in terms of the realized channel when we limit ourselves to each coding scheme and use powerful coding. We examine the cases where successive cancellation based decoding is used and where optimal joint decoding is done.

The main idea is that we can consider the two independent codewords in each of the designs proposed as belonging to two different users, each trying to get some information through, but suffers interference from the other user. This turns out to fit the well-studied multiple access channel (MAC) framework, the capacity theorem for which is [7]:

**Theorem 2** (*Multiple access channel capacity*): *The capacity of a multiple access channel  $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2),)$  is the closure of the convex hull of all  $(R_1, R_2)$  satisfying*

$$R_1 < I(X_1; Y | X_2) \quad (5.17)$$

$$R_2 < I(X_2; Y | X_1) \quad (5.18)$$

$$R_1 + R_2 < I(X_1, X_2; Y) \quad (5.19)$$

for some product distribution  $p_1(x_1)p_2(x_2)$  on  $\mathcal{X}_1 \times \mathcal{X}_2$ .

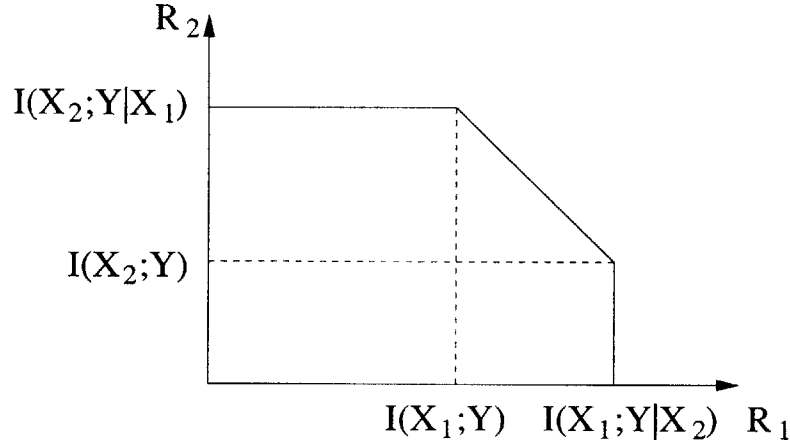


Figure 5-15: Capacity region for a multiple access channel.

A typical achievable rate region associated with a certain input distribution is draw in Figure 5-15. The diagonal corresponds to the bound

$$R_1 + R_2 < I(X_1, X_2; Y) \quad (5.20)$$

$$= I(X_1; Y) + I(X_2; Y|X_1) \quad (5.21)$$

$$= I(X_2; Y) + I(X_1; Y|X_2) \quad (5.22)$$

$$= I(X_1; Y) + I(X_2; Y) + I(X_1; X_2|Y) \quad (5.23)$$

The various expressions can be shown to be equivalent with simple manipulation. When this bound is achieved, it is as if the two users are behaving as a single user. There is no loss for doing encoding separately.

The achievable rate region is a function of both the input distribution used,  $p_1(x_1)$ ,  $p_2(x_2)$ , and the channel statistics,  $p(y|x_1, x_2)$ . The rate pair  $(R_1, R_2)$  depends on the number of codewords chosen at the transmitter, and might be inside, on the edge of, or outside the achievable rate region.

If the transmitter knows the channel, then it can maximize performance by choosing to transmit at rates just inside of the boundary, and even change the input distribution  $p_1(x_1)$  and  $p_2(x_2)$ , to modify the achievable rate region.

On the other hand, when the transmitter has no knowledge of the channel, which is the case we are studying, the above can not be done. Instead, the transmitter would use a fixed input distribution and rate pair  $(R_1, R_2)$ . When the realized channel is weak, such that the achievable rate region does not include the rate pair, we are in outage. When the realized channel is strong enough, the rate pair is achievable theoretically using infinitely powerful codes. In this case, depending on where  $(R_1, R_2)$  is inside the region, joint decoding might or might not be required. This is depicted in Figure 5-16.

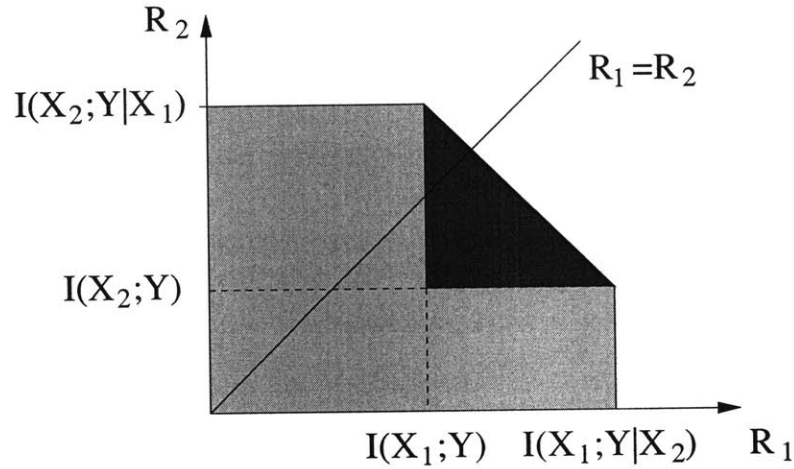


Figure 5-16: The achievable rate region has two sub-regions. The darkly shaded one requires joint decoding.

When  $(R_1, R_2)$  is inside the lightly shaded region, joint decoding is not required. For example, if  $R_1 < I(X_1; Y)$ , then we can decode the message from user 1 first, treating user 2 as noise, and then decode the message from user 2 after canceling out user 1. This exploits the mutual information chain rule in (5.21) and (5.22). This is the successive cancellation idea used in BLAST. From now on, we will refer to this type of decoding as separate decoding (as opposed to joint decoding).

When  $(R_1, R_2)$  is inside the darker region, decoding is more difficult. Joint decoding using ML or typicality decoder can be used, but are quite complex. Other ways to achieve rate pairs inside the darker region include time sharing or rate splitting [30] at the transmitter. The problem is that they require the transmitter to know



where the corner points of the achievable rate region are.

To further study what rates can be achieved when the transmitter does not have knowledge of the achievable rate region, let us restrict the rates to  $R_1 = R_2 = R/2$ , i.e., symmetric between the two users.<sup>3</sup> Now we can look at the outage conditions in the two cases of joint and separate coding.

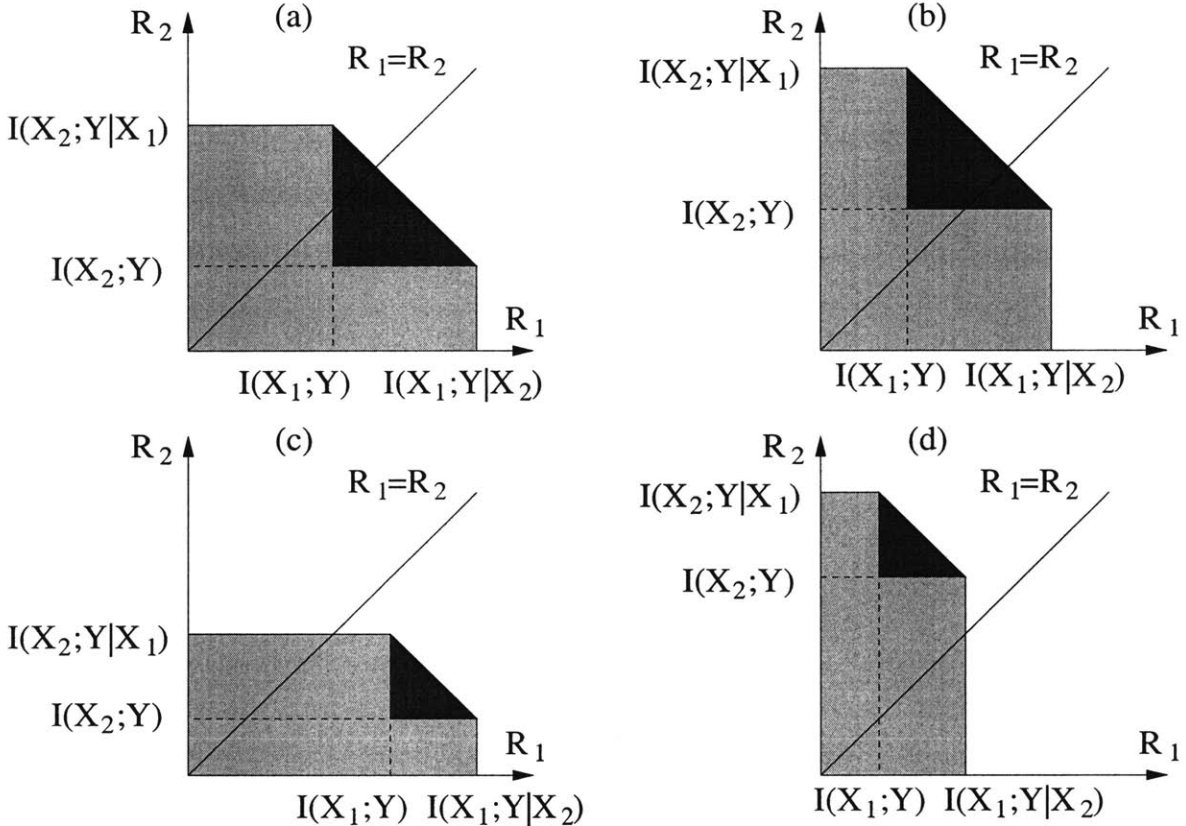


Figure 5-17: The four different ways in which the  $R_1 = R_2$  line can intersect the achievable rate bounds.

Depending on the channel realized, the achievable rate region takes on different shapes. In terms of how the  $R_1 = R_2$  line intersects the achievable rate bounds, there are four different cases, as shown in Figure 5-17, (a)-(d). For the two cases of joint and separate decoding, the maximum rate  $R$  that can be supported is tabulated in

<sup>3</sup>In one variation of V-BLAST, using different rates for the two codes was considered [28]. They fixed the decoding order and assigned lower rate for the code decoded first. Here, we restrict ourselves to symmetric cases.

Table 5.1. We see that for cases (c) and (d), doing joint decoding or not does not

Table 5.1: Rates achievable when joint decoding and separate decoding are used in various cases depicted in Figure 5-17.

	Joint	Separate
(a)	$I(X_1, X_2; Y)$	$2I(X_1; Y)$
(b)	$I(X_1, X_2; Y)$	$2I(X_2; Y)$
(c)	$2I(X_2; Y X_1)$	$2I(X_2; Y X_1)$
(d)	$2I(X_1; Y X_2)$	$2I(X_1; Y X_2)$

matter. For cases (a) and (b), there is a difference, and could potentially be vary large.

In this section, using the multiple access channel framework, we examined the maximum rate achievable when there are two independent codewords, or equivalently, two users, with the same rate. Let us express the bounds listed in Table 5.1 for all four cases together in a more concise form. With joint decoding, the achievable rate is

$$R < \min(I(X_1, X_2; Y), 2I(X_1; Y|X_2), 2I(X_2; Y|X_1)). \quad (5.24)$$

When joint decoding is unfeasible, using separate decoding based on successive cancellation, the achievable rate is

$$R < 2 \max(\min(I(X_1; Y), I(X_2; Y|X_1)), \min(I(X_2; Y), I(X_1; Y|X_2))). \quad (5.25)$$

In the next three sections, we use this tool to analyze each of the three designs proposed in section 5.4.1. Let us assume the input distribution is IID complex Gaussian with SNR per antenna  $\rho$ . For the two cases of using joint decoding and separately decoding based on successive cancellation, we express the rates achievable explicitly as functions of the realized channel  $\mathbf{H}$ , and evaluate the diversity-multiplexing trade-offs achieved. We also plot families of outage probability curves, and compare to that of the ultimate limit corresponding to the channel outage probability.

### 5.4.3 V-BLAST

In this section, we focus on the specific case of using V-BLAST encoding on a  $2 \times 2$  multiple antenna channel with Rayleigh fading, which was described in Figure 5-12.

Let us first evaluate the rates achievable using joint and separate decoding using (5.24) and (5.25) respectively.

To evaluate  $I(X_1, X_2; Y)$ , let us pretend joint encoding were used instead of two independent codeword. Then, the capacity achievable is just the channel capacity,

$$I(X_1, X_2; Y) = \log_2(1 + \rho\|\mathbf{H}\|^2 + \rho^2|\det(\mathbf{H})|^2). \quad (5.26)$$

To evaluate  $I(X_1; Y|X_2)$  and  $I(X_2; Y|X_1)$ , we need to look at what each user can achieve after the other user has been canceled out completely. Without interference from user 2, user 1 sees an effective channel gain of  $\|\mathbf{h}_1\|^2$ , where  $\mathbf{h}_1$  is the first column of  $\mathbf{H}$ . Similarly, user 2 experiences  $\|\mathbf{h}_2\|^2$ . Therefore,

$$I(X_1; Y|X_2) = \log_2(1 + \rho\|\mathbf{h}_1\|^2), \quad (5.27)$$

$$I(X_2; Y|X_1) = \log_2(1 + \rho\|\mathbf{h}_2\|^2). \quad (5.28)$$

Substituting the above into (5.24), we obtain the outage condition for joint decoding, which is,

$$R > \min(\log_2(1 + \rho\|\mathbf{H}\|^2 + \rho^2|\det(\mathbf{H})|^2), 2\log_2(1 + \rho\|\mathbf{h}_1\|^2), 2\log_2(1 + \rho\|\mathbf{h}_2\|^2)). \quad (5.29)$$

Focusing on just the last term, a lower bound on the outage probability is

$$P_{\text{out}} > P[2\log_2(1 + \rho\|\mathbf{h}_2\|^2) < R] \doteq P[\rho\|\mathbf{h}_2\|^2 < \rho^{r/2}], \quad (5.30)$$

where we use  $2^R \sim \rho^r$ . Since  $\|\mathbf{h}_2\|^2$  is a chi-squared random variable of order 4,  $P_{\text{out}} \geq P[\|\mathbf{h}_2\|^2 < \rho^{r/2-1}] \doteq \rho^{r-2}$ . Thus, an upper bound on the diversity-multiplexing tradeoff achieved is  $d(r) = 2 - r$ , a straight line between  $(0, 2)$  and  $(2, 0)$ . This is drawn as a solid line in Figure 5-18.

For separate decoding based on successive cancellation, the outage condition can be obtained using (5.25). From there, it can be shown that an upper bound on the diversity-multiplexing tradeoff achievable is  $d(r) = 1 - r/2$  [41], i.e., a straight line between  $(0, 1)$  and  $(2, 0)$ , which is drawn as a dashed line in Figure 5-18. The main intuition for the maximum diversity gain to be only 1 is that every time the channel is near singular, i.e.,  $\rho^2 |\det(\mathbf{H})|^2 < 1$ ,  $I(X_1, X_2; Y)$  becomes similar to  $I(X_1; Y|X_2)$  and  $I(X_2; Y|X_1)$ . Thus,  $I(X_1; Y)$  and  $I(X_2; Y)$  would be small, so the right-hand side of (5.25) is small.

Comparing to the optimal diversity-multiplexing tradeoff achievable by  $2 \times 2$  systems (shown in Figure 2-4), V-BLAST achieves the maximum multiplexing gain of 2, but not the maximum diversity gain of 4. The diversity gain loss is more severe when joint decoding can not be used.

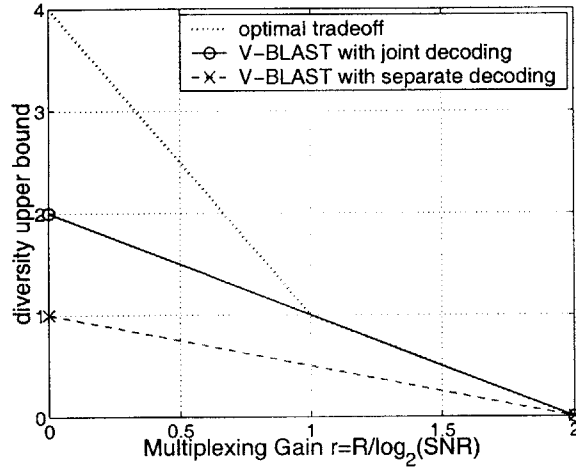


Figure 5-18: Diversity-multiplexing tradeoffs achieved by V-BLAST encoding with joint and separate decoding.

Next, let us compare the outage probabilities achieved by V-BLAST to that of the ultimate channel outage probability limit, for the two cases of joint and separate decoding. The families of outage probability curves are plotted as thick dashed lines in Figure 5-19 and Figure 5-20, respectively. The corresponding channel outage probability curves are plotted as thin solid lines.

We see that when joint decoding is used (Figure 5-19), the limiting slope of each

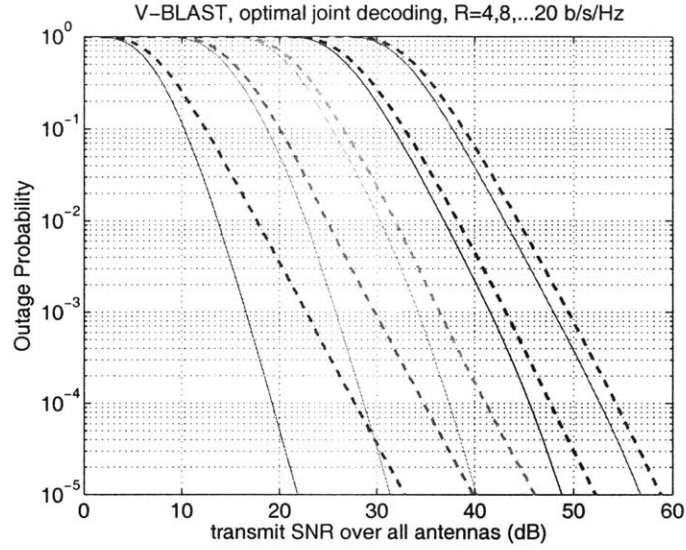


Figure 5-19: Outage probability curves for rates  $2, 4, \dots, 20$  b/s/Hz achieved by V-BLAST encoding with joint decoding (thick dashed), comparing with that of channel outage probability (thin solid).

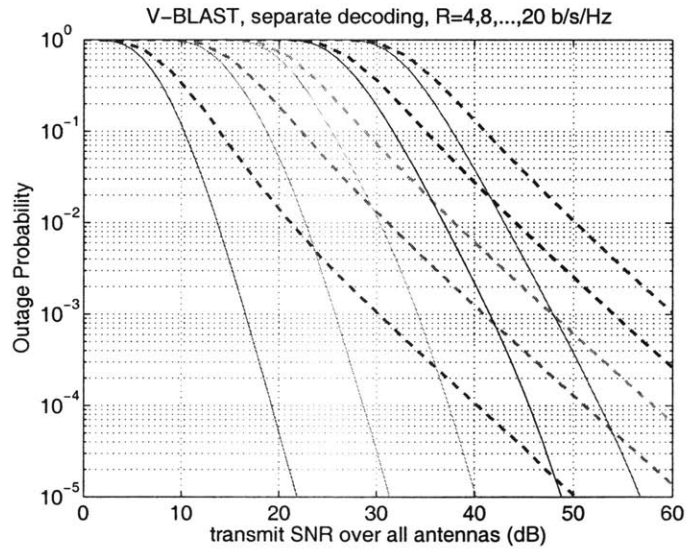


Figure 5-20: Outage probability curves for rates  $2, 4, \dots, 20$  b/s/Hz achieved by V-BLAST encoding with more practical separate decoding based on successive cancellation (thick dashed), comparing with that of channel outage probability (thin solid).

curve is only 2, which is sub-optimal compared to the channel outage probability curves. However, for higher rates, this deficiency does not have a significant consequence. V-BLAST with joint decoding is only 1 to 2 dB from optimal at  $10^{-3}$  target outage probability.

When separate decoding is used (Figure 5-20), the limiting slopes of each curve is only 1, and there is a significant discrepancy between the outage probability curves achieved and the performance limit. At  $10^{-3}$  target outage probability, the gaps between the corresponding solid and dashed curves are over 10 dB, and become even larger for lower target outage probabilities.

We note that V-BLAST achieves the maximum multiplexing gain when either joint decoding or separate decoding is use. Therefore, the spacings between the curves for both cases are 2 bits per 3 dB, which is optimal. Therefore, for a fixed target error rate, the gaps to the optimal curves approach constant values as rate increases.

#### 5.4.4 Two-Layer-D-BLAST

Now let us turn to the two-layer-D-BLAST design illustrated in Figure 5-13. Compared to the V-BLAST design we just studied, there are two main differences.

The first one is that this code effectively has three different segments as we can see visually in Figure 5-13. Because of this, the mutual information achieved is the average of the three region. We have,

$$I(X_1, X_2; Y) = \frac{1}{3} \left( \log_2(1 + \rho \|\mathbf{h}_1\|^2) + \log_2(1 + \rho \|\mathbf{H}\|^2 + \rho^2 |\det(\mathbf{H})|^2) + \log_2(1 + \rho \|\mathbf{h}_2\|^2) \right), \quad (5.31)$$

where the three terms corresponds to the left, middle, and right regions.

The other feature of this design, which is absent in V-BLAST, is the symmetry between the two users in terms of the individual mutual information achieved. Also using averaging, we have

$$I(X_1; Y|X_2) = I(X_2; Y|X_1) = \frac{1}{3} \left( \log_2(1 + \rho \|\mathbf{h}_1\|^2) + \log_2(1 + \rho \|\mathbf{h}_2\|^2) \right). \quad (5.32)$$

Combining the above two equations and using the chain rule, we have,

$$I(X_1; Y) = I(X_2; Y) = \frac{1}{3} \log_2(1 + \rho \|\mathbf{H}\|^2 + \rho^2 |\det(\mathbf{H})|^2). \quad (5.33)$$

As a consequence of the symmetry, we have the following chain of inequalities:

$$I(X_1; Y) = I(X_2; Y) \leq \frac{I(X_1, X_2; Y)}{2} \leq I(X_1; Y|X_2) = I(X_2; Y|X_1). \quad (5.34)$$

Using this inequality, the upper bound on the rate achievable when joint decoding is used stated in (5.24) becomes

$$R < I(X_1, X_2; Y). \quad (5.35)$$

This means there is no loss for having two separate codewords when joint decoding is used. And the bound for using separate decoding stated in (5.25) becomes

$$R < 2I(X_1; Y) = \frac{2}{3} \log_2(1 + \rho \|\mathbf{H}\|^2 + \rho^2 |\det(\mathbf{H})|^2) = \frac{2}{3} C_{\text{channel}}(\mathbf{H}). \quad (5.36)$$

Now let us evaluate the diversity-multiplexing tradeoff achieved in the two different cases. When separate decoding is used, the maximum rate achievable is simply 2/3 of the channel capacity. Therefore, the tradeoff achieved is the optimal tradeoff scaled by 2/3 in the multiplexing gain direction, drawn as a dashed line in Figure 5-21.

When joint decoding is used, the outage probability is

$$P_{\text{out}} = P[I(X_1, X_2; Y) < R] \quad (5.37)$$

$$\doteq P[(1 + \rho \|\mathbf{h}_1\|^2)(1 + \rho \|\mathbf{H}\|^2 + \rho^2 |\det(\mathbf{H})|^2)(1 + \rho \|\mathbf{h}_2\|^2) < \rho^{3r}] \quad (5.38)$$

$$< P[\rho^4 \|\mathbf{h}_1\|^2 |\det(\mathbf{H})|^2 \|\mathbf{h}_2\|^2 < \rho^{3r}] \quad (5.39)$$

In (5.39), the upper bound is obtained by keeping only the highest order term. To evaluate (5.39), we use the technique used in section 2.3.3. Let us change basis by performing a QR decomposition on  $\mathbf{H}$ . Now we have  $|\det(\mathbf{H})|^2 = r_{11}^2 r_{22}^2$ ,  $\|\mathbf{h}_1\|^2 = r_{11}^2$ ,

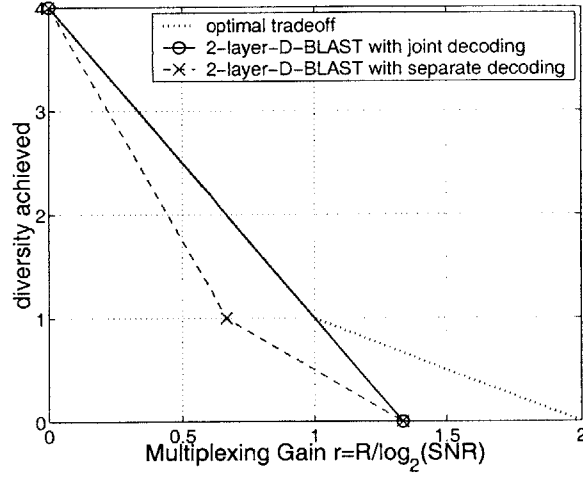


Figure 5-21: Diversity-multiplexing tradeoffs achieved by two-layer-D-BLAST encoding with joint and separate decoding.

and  $\|\mathbf{h}_2\|^2 = |r_{12}|^2 + r_{22}^2 \geq |r_{12}|^2$ . Recall that  $r_{11}^2$  is a chi-squared random variable of order 4,  $|r_{12}|^2$  and  $r_{22}^2$  are chi-squared random variables of order 2, and they are all independent. Therefore

$$P_{\text{out}} < P[\|\mathbf{h}_1\|^2 |\det(\mathbf{H})|^2 \|\mathbf{h}_2\|^2 = r_{11}^4 r_{22}^2 |r_{12}|^2 < \rho^{3r-4}] \doteq \rho^{3r-4}. \quad (5.40)$$

This corresponds to a lower bound on the diversity-multiplexing tradeoff curve of  $d(r) = 4 - 3r$ , which is a straight line between  $(0, 4)$  and  $(4/3, 0)$  and drawn as a solid line in Figure 5-21. For  $0 \leq r \leq 1$ , this lower bound meets the optimal tradeoff upper bound achievable by any system. Therefore, the bound must be tight. For  $r > 1$ , we need to look at (5.38). In this range,  $3r > 3$ . Therefore, if (5.38) is expanded, then the only term that matters is the highest order term we kept in (5.39). Thus, the tradeoff bound is exact.

We see that this two-layer-D-BLAST design achieves the maximum diversity gain, but not the maximum multiplexing gain. This is similar to OSTBC, which was studied in section 5.2.

We now evaluate numerically the outage probabilities achieved when joint and separate decoding are used. The families of outage probability curves are plotted as



thick dashed lines in Figure 5-22 and Figure 5-23, respectively. The corresponding channel outage probability curves are plotted as thin solid lines for comparison.

Comparing to the optimal curves, we see that the outage probabilities achieved by the two-layer-D-BLAST design have the same limiting slope, i.e., the same maximum diversity. However, due to the sub-optimal multiplexing gain, the gap to optimality, i.e., the gap between the corresponding solid and dashed curves grows indefinitely. Therefore, this design should not be used at high SNR. Recall that this phenomenon was also exhibited by OSTBC in Figure 5-3.

It is also worth noting that there is not a significant difference between using joint decoding and using separate decoding. In the sense that they both have large gaps at high SNR and non-zero gaps at low SNR.

### 5.4.5 X-BLAST

The last design to be analyzed is the X-BLAST design shown in Figure 5-14. This design has complete symmetry between the two users. Using similar techniques as in the previous cases, it can be shown that

$$I(X_1, X_2; Y) = \log_2(1 + \rho\|\mathbf{H}\|^2 + \rho^2|\det(\mathbf{H})|^2), \quad (5.41)$$

$$I(X_1; Y|X_2) = I(X_2; Y|X_1) = \frac{1}{2} (\log_2(1 + \rho\|\mathbf{h}_1\|^2) + \log_2(1 + \rho\|\mathbf{h}_2\|^2)). \quad (5.42)$$

Also due to symmetry, we have the same chain of inequalities as in (5.34). When joint decoding is used, the maximum rate achievable is also  $I(X_1, X_2; Y)$ , which is the same as the channel capacity for this design. Therefore, there is no loss for using X-BLAST encoding when joint decoding is used.

When separate decoding is used, we have a situation similar to the V-BLAST case. The quantity  $I(X_1; Y|X_2) + I(X_2; Y|X_1)$  equals  $\log_2(1 + \rho\|\mathbf{h}_1\|^2) + \log_2(1 + \rho\|\mathbf{h}_2\|^2)$  in both cases. The quantity  $I(X_1; Y) + I(X_2; Y)$  from both cases are also equal. Together with the achievable rate upper bound in (5.25), these lead us to believe that the diversity-multiplexing tradeoff achieved by X-BLAST with separate decoding is the same as that by V-BLAST, which is  $d(r) = 1 - r/2$ , a straight line between  $(0, 1)$

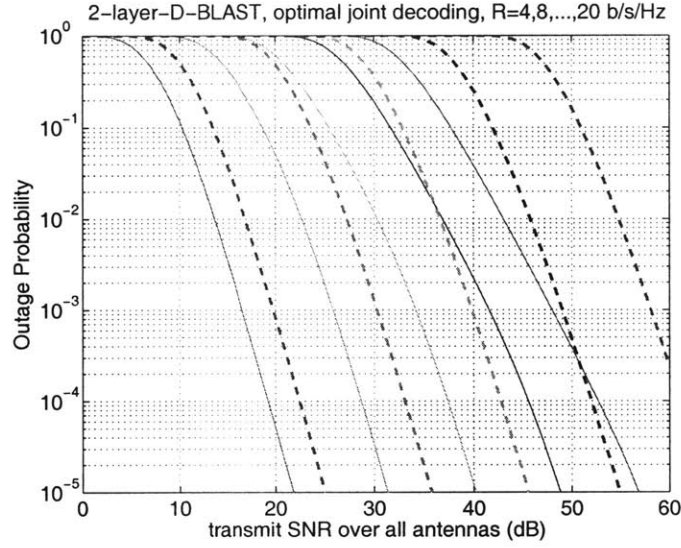


Figure 5-22: Outage probability curves for rates  $2, 4, \dots, 20$  b/s/Hz achieved by two-layer-D-BLAST encoding with joint decoding (thick dashed), comparing with that of channel outage probability (thin solid).

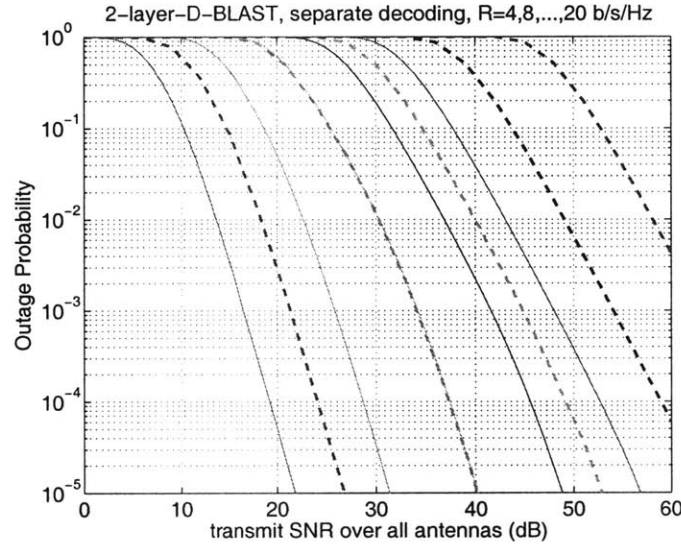


Figure 5-23: Outage probability curves for rates  $2, 4, \dots, 20$  b/s/Hz achieved by two-layer-D-BLAST encoding with more practical separate decoding (thick dashed), comparing with that of channel outage probability (thin solid).

and  $(2, 0)$ .

The tradeoff curves for both cases are drawn in Figure 5-24. We see that using joint decoding yields the optimal tradeoff, while using separate decoding results in significant loss in diversity gain, similar to V-BLAST.

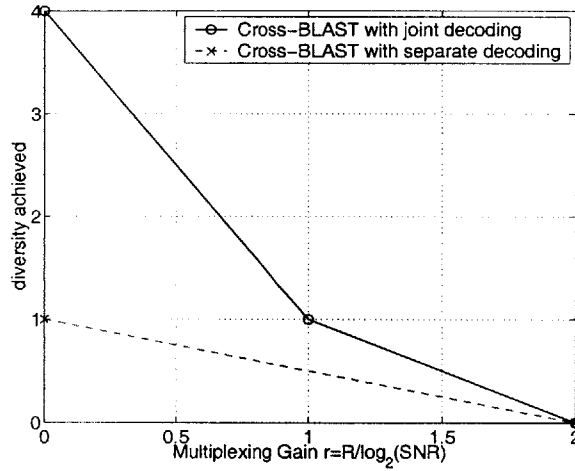


Figure 5-24: Diversity-multiplexing tradeoffs achieved by X-BLAST encoding with joint and separate decoding.

We also evaluate numerically the outage probabilities achieved by X-BLAST when joint decoding and separate decoding is used. The families of outage probability curves are plotted as thick dashed lines in Figure 5-25.

When joint decoding is used, the optimal outage probability is achieved. When separate decoding is used, the resulting outage probability curves are similar to that of V-BLAST with separate decoding plotted earlier in Figure 5-20. The curves approximately form a set of parallel lines with slope 1 and gap 2 bits per 3 dB. This implies a diversity-multiplexing tradeoff of  $d(r) = 1 - r/2$ , which is what we believe.

### 5.4.6 Comparisons

In this section, we look collectively at the performance achieved by the various systems we evaluated in the last three sections, as well as the OSTBC studied in section 5.2. The goal is to identify which scheme is better, and in which SNR regime. This could

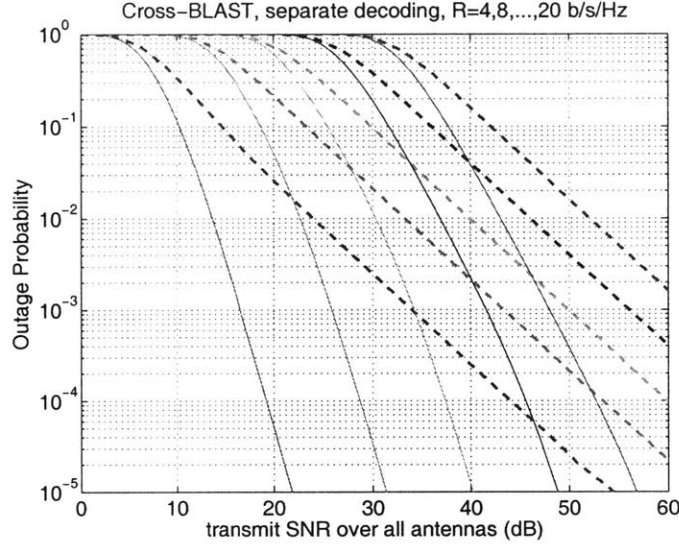


Figure 5-25: Outage probability curves for rates  $2, 4, \dots, 20$  b/s/Hz achieved by X-BLAST encoding with more practical separate decoding (thick dashed), comparing with doing joint decoding, which is also the channel outage probability (thin solid).

help us identify areas that are sufficiently solved and areas that require more effort. We also want to see how the diversity-multiplexing tradeoff achieved affect the actual performance in the regime we care about. This could lead to design criteria that are specific for different regimes.

In the previous sections, we plotted families of outage probability curves. To summarize and compare them, we transform each family of curves to one curve of gap to capacity (in dB) vs. rate. More specifically, we measure the gaps between the dashed lines and the corresponding solid ones, for all rates, at target outage probability  $10^{-3}$ .<sup>4</sup> The resulting curves for the seven different cases are plotted in Figure 5-26. The cases requiring joint decoding are drawn as dashed lines, indicating that they are currently still unimplementable.

In Figure 5-26, the lower the curve is the better. Next, we go through various SNR and rate regimes, compare the performance of different schemes in that regime, and identify which scheme is the best.

<sup>4</sup>Throughout the rest of the discussion, we assume that we are always interested in  $10^{-3}$  (long block) error rate. The numerical values quoted would vary otherwise.

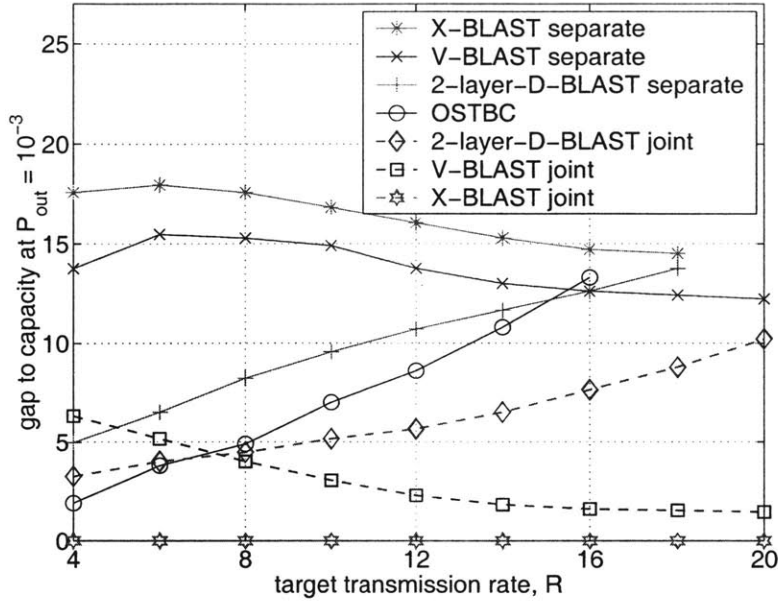


Figure 5-26: Gaps to capacity as a function of rate at  $P_{\text{out}} = 10^{-3}$  for various systems.

- First of all, X-BLAST with joint decoding is optimal for all rates and SNR.<sup>5</sup> Therefore, if joint decoding can somehow be managed efficiently, we should use this scheme.
- Below  $R = 6$  b/s/Hz (SNR = 20 dB), OSTBC is near optimal, less than 4 dB away, and very efficient. Therefore, in the low SNR regime, the  $2 \times 2$  case is essentially solved. There is very little room for improvement.
- Between  $R = 6$  b/s/Hz and 16 b/s/Hz (SNR between 20 dB and 55 dB),
  - OSTBC is the best among the currently implementable schemes listed (solid lines). However, it is up to 12 dB away from capacity.
  - Using joint decoding can significantly improve the performance.
  - Two-layer-D-BLAST with joint decoding might be the simplest to implement; some iterative decoding method might succeed. However, it provides the least improvement.

<sup>5</sup>The only loss might be an increase in delay required to achieve the same error probability, which is common for all the schemes we are looking at in this section.

- V-BLAST with joint decoding provide further gain. However, implementation might be as difficult as the X-BLAST case.
- Above  $R = 16$  b/s/Hz (SNR = 55 dB),
  - V-BLAST with separate decoding based on successive cancellation is the best among the currently implementable schemes listed. It is about a constant 12 dB away from capacity.
  - OSTBC and the two-layer-D-BLAST design become very far from optimal in the high rate regime due to the loss of multiplexing gain.

Besides the families of outage probability curves, we also evaluate the diversity-multiplexing tradeoff curves achieved. They are collected in Figure 5-27.

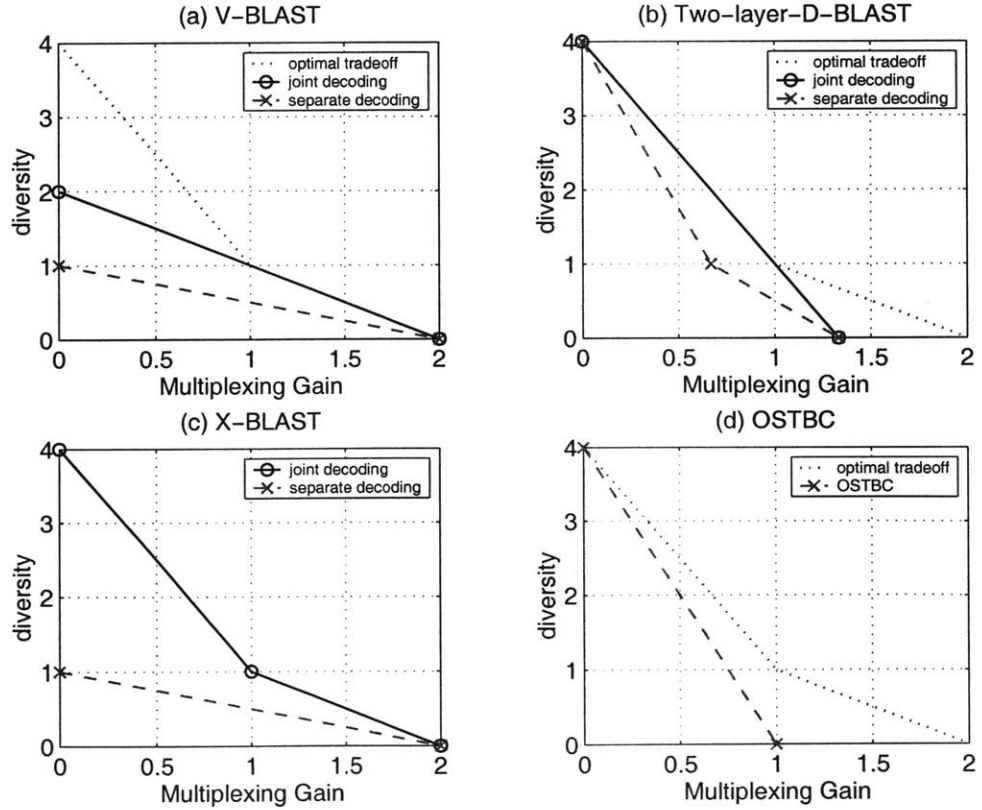


Figure 5-27: Diversity-multiplexing tradeoff curves achieved by variations of BLAST in block form and OSTBC.

When joint decoding is used (solid curves), X-BLAST achieves the optimal trade-off, V-BLAST achieves only the lower segment of the optimal tradeoff, while two-layer-D-BLAST achieves only the upper segment. With separate decoding, X-BLAST and V-BLAST achieves only the maximum multiplexing gain point, while two-layer-D-BLAST and OSTBC achieves only the maximum diversity gain point.

Looking at Figure 5-26 and Figure 5-27 together, we can get some ideas of how diversity-multiplexing tradeoff achieved can affect the actual performance in the regime we care about. This could lead to design criteria that are specific for different regimes.

In the low SNR, low rate regime, it seems that achieving the maximum diversity gain is important. This can be realized by the design criterion of maximizing the worst case determinant, or at least keeping it away from zero.

In the high SNR, high rate regime (high relative to a certain target error rate), it appears that achieving the maximum multiplexing gain is important, so that the SNR required would grow with rate as slowly as possible. This can be realized by utilizing all degrees of freedom and not using repetition. Comparing V-BLAST with joint and with separate decoding, we see that it is much more advantageous to achieve the entire lower segment of the tradeoff curve. The design criterion for this is still unclear.

In the next two sections, we explore a couple schemes that actually employ joint decoding, instead of successive cancellation based separate decoding. In particular, we look at systems where tilted-QAM codes proposed in section 4.3 are combined with hard-decision and soft-decision based error correction codes.

## 5.5 Tilted-QAM With Hard-Decision ECC

In this section, we build upon the tilted-QAM code we developed earlier in Chapter 4, which achieves the optimal diversity-multiplexing tradeoff, and will strengthen it with powerful error correction coding to obtain additional coding gain. In particular, we concatenate it with a Reed-Solomon (RS) outer code with hard-decision based

decoding. RS code is a commonly used hard-decision code that has been used in industry for decades.

We first describe the system setup and then present simulation results. We show that combining tilted-QAM with a hard-decision ECC can reach about 5 dB from capacity with moderate complexity.

### 5.5.1 System Setup

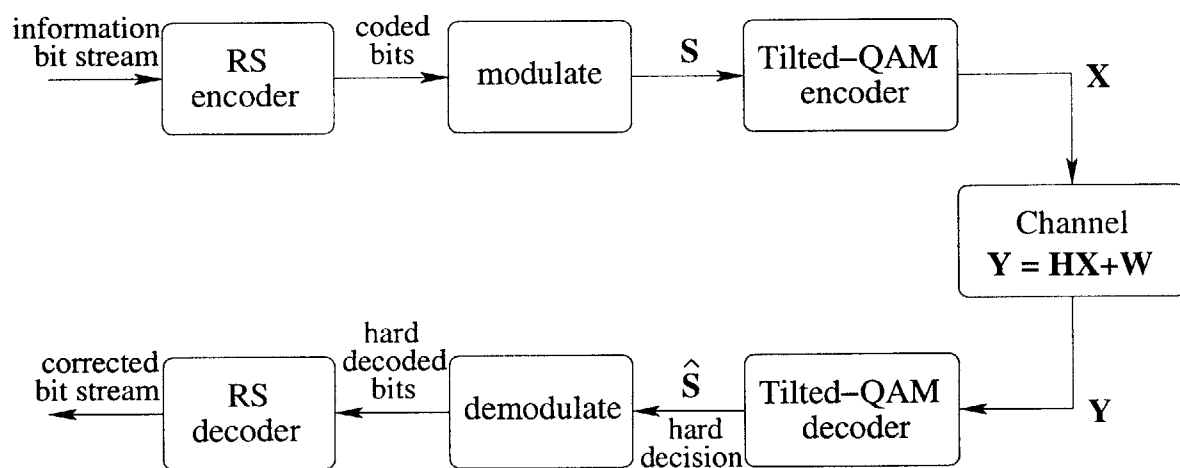


Figure 5-28: Concatenation of a tilted-QAM inner code with a Reed-Solomon outer code.

The tilted-QAM-Reed-Solomon concatenated code system is depicted in Figure 5-28. The tilted-QAM code is used as the inner code around the channel to transform it from a multiple antenna channel to a generic channel with certain error rate. The RS code is then used as an outer code to provide additional redundancy; if errors occur across the effective inner channel, they can be corrected.

At the encoder, information bits are first encoded with an RS code. In particular, we choose to use GF(256) RS code, so that each RS symbol conveniently corresponds to one byte<sup>6</sup>. The encoded bits are then modulated into symbols using Gray-labeling,

<sup>6</sup>In RS coding, groups of bits are mapped to RS symbols from an algebraic field, for example, GF(256). The redundancy or coding is introduced at the symbol level.



as depicted earlier in Figure 5-9. Every eight (real) symbols,  $\mathbf{S}$ , are then encoded into one  $2 \times 2$  block according to the tilted-QAM coding scheme before transmission.

A particular detail of the implementation is that during the modulation step, we choose to modulate the eight bits that belong to one RS symbol to the same bit level (eg. MSB, LSB) within one  $2 \times 2$  block. The motivation is to reduce the number of RS symbol errors resulting from bit errors, by grouping bits that experience similar error rates and are somewhat correlated into one RS symbol.

At the decoder, the reverse of the encoding steps are performed. First, for each  $2 \times 2$  block of received symbols,  $\mathbf{Y}$ , a sphere decoder is used to find the most likely eight symbols  $\hat{\mathbf{S}}$  that would result in  $\mathbf{Y}$ , dealing with the combined effect of the channel and the tilted-QAM encoder. Once all the symbols in a codeword are detected, they are demodulated and put through an RS decoder to correct any error that might have occurred. If the number of byte errors is less than what the RS code can tolerate, then the decoding would be successful.

We stress that the decoding is hard-decision based. Information regarding how close  $\mathbf{Y}$  is to the constellation points is disregarded. It is well-known that doing hard-decision is sub-optimal, although it is often used to reduce complexity.

### 5.5.2 Simulation Results

Using the system setup described above, we perform three groups of simulations in the moderate to high SNR regime using 16-QAM, 64-QAM, and 256-QAM constellations. The RS code-length used is 256 bytes (255 for 64-QAM) <sup>7</sup>. To evaluate the effect of coding and to gauge how much coding should be used, a series of experiments are performed for each constellation with increasingly stronger RS codes to correct increasingly more errors. In other words, data rate is reduced in exchange for better performance. In the 16-QAM case, five experiments are performed with code-rates 1 (uncoded), 7/8, 3/4, 5/8, and 1/2. These code-rates are chosen so that the data rate would conveniently correspond to  $R = 8, 7, 6, 5, 4$  b/s/Hz. Similarly, in the 64-QAM

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<sup>7</sup>The 256 bytes, or 2048 bits, is the length of the RS codeword, not the length of the uncoded message.

case, seven experiments are run with  $R = 12, 11, \dots, 6$  b/s/Hz; in the 256-QAM case, there are nine experiments with  $R = 16, 15, \dots, 8$  b/s/Hz.

The resulting block error rate curves are shown in Figure 5-29. The three groups of experiments can be identified by the different SNR intervals and the experiments within the groups are selectively labeled by the rates  $R$ .

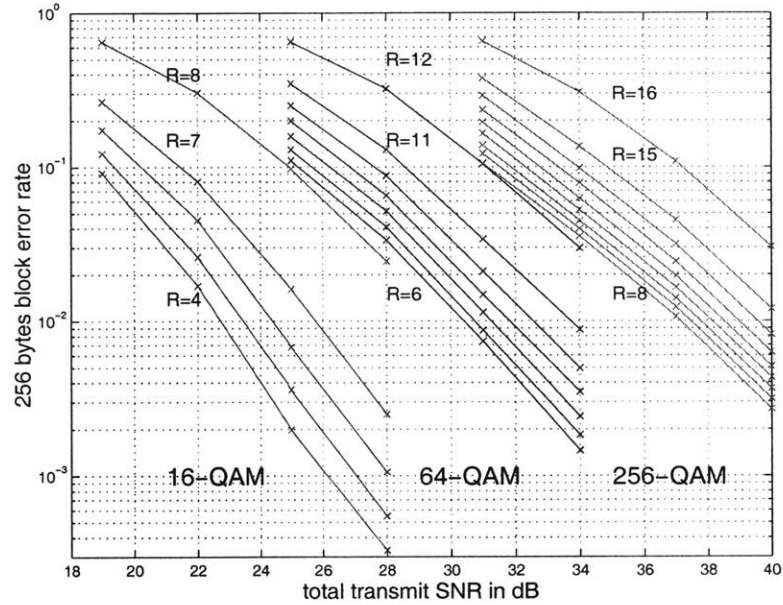


Figure 5-29: Block error rate curves for 16-QAM, 64-QAM, and 256-QAM cases. As we gradually reduce the data rate by 1 b/s/Hz, the block error rate lowers due to stronger coding. However, the gain diminishes.

We see that for each of the three groups of curves, the top gap is always the largest, which corresponds to the greatest improvement in block error rate for 1 bit of rate reduction. As we reduce the rate further, although the block error rate improves, the gain diminishes. The reason for the diminishing gain is that we are operating in the regime where the uncoded error rate is not too high. Although there are often some errors to be corrected, most of those times, there are only a few errors. It becomes less likely to have more.

In conclusion, it appears that a small amount of coding is sufficient, about 1 b/s/Hz rate reduction. If we apply too much coding, then the gain from correcting a few more block errors would not be worth the reduction in rate, or the constellation

expansion.

Next, let us look at how close the tilted-QAM-RS concatenated coding scheme can approach the capacity limit. The block error rate curves for the  $R = 7$  (16-QAM),  $R = 11$  (64-QAM), and  $R = 15$  (256-QAM) cases are re-plotted in Figure 5-30, together with the corresponding outage probability curves.

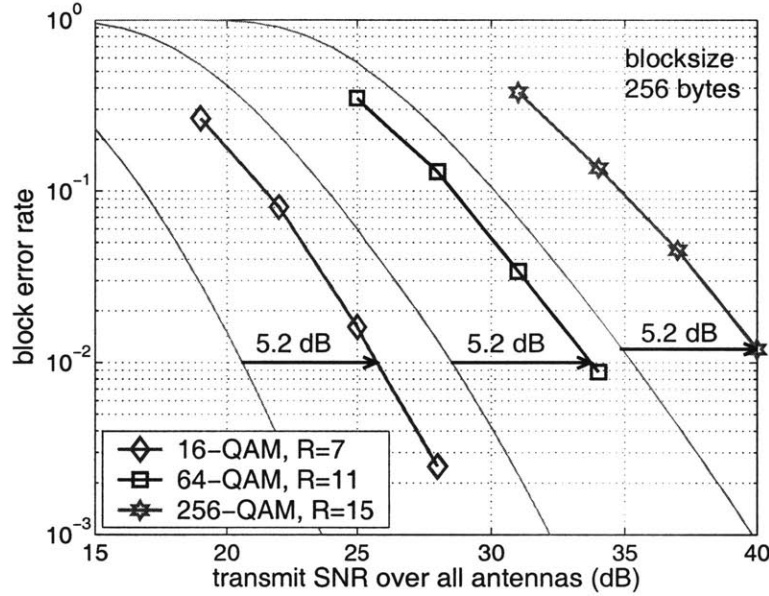


Figure 5-30: Block error rate curves for 16-QAM, 64-QAM, and 256-QAM cases with 1 b/s/Hz rate reduction using RS coding. The unmarked curves are the corresponding channel outage probability curves.

At  $10^{-2}$  block error rate, we see that 5.2 dB gaps to the ultimate performance limits are achieved for all three rates. This is slightly worse than the D-BLAST-MMSE, which achieves 4.8 dB gaps at  $10^{-2}$  for  $R = 6$  and 8 b/s/Hz.

## 5.6 Tilted-QAM with Soft-Decision ECC

In this section, we look at how to enhance tilted-QAM codes with soft-decision based error correction codes, in particular, low-density-parity-check (LDPC) codes, which is a powerful soft-decision code that can approach close to capacity. Other soft decision codes, such as turbo codes, may also be used. We simply choose to work with LDPC

here.

Soft-decision decoding is generally believed to be better performing, although potentially more complex, than hard-decision decoding. In the soft-decision case, bits decoded are assigned confidence measures. Bits that are incorrectly decoded often have lower confidence measures than the correctly decoded ones. Soft decision decoding takes advantage of this difference to allow correction of more bit errors.

We first present the system setup, then describe an iterative soft-decision decoding procedure in detail, and finally present simulation results. We show that the proposed system can reach about 3 dB from capacity.

### 5.6.1 System Setup

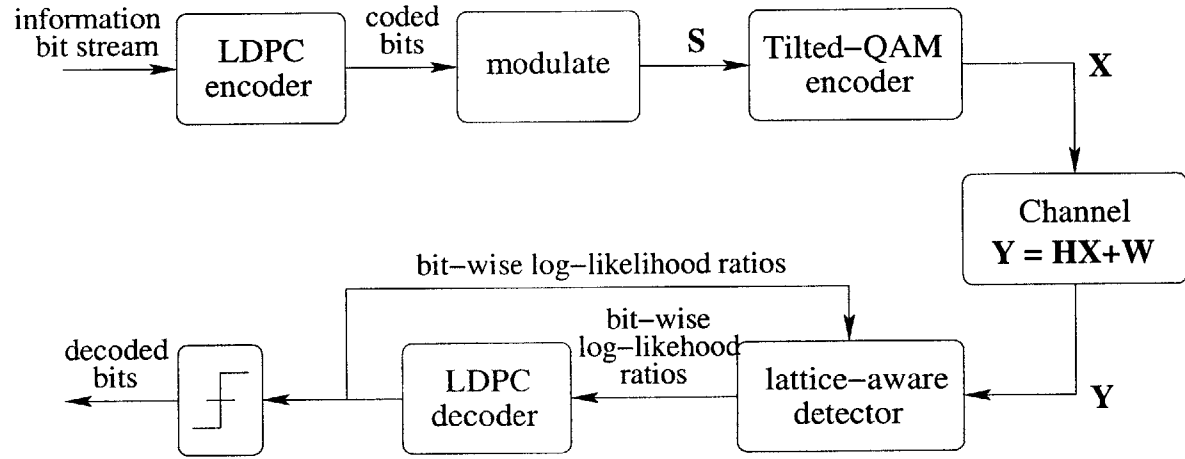


Figure 5-31: Concatenation of a tilted-QAM inner code with an LDPC outer code with a two component iterative soft-decision decoder.

A tilted-QAM-LDPC concatenated code system is depicted in Figure 5-31. The concatenated structure of the encoder is very similar to that used in the hard-decision case. Information bits are first encoded using an LDPC code, modulated into symbols, and then encoded into  $2 \times 2$  blocks according to the tilted-QAM coding scheme before transmission.

One implementation detail we briefly mention here is that an interleaver is used at the output of the LDPC to scramble the coded bits before modulation. The

motivation is that bits modulated within the same  $2 \times 2$  tilted-QAM coded matrix suffer correlated errors. We want those bits to be well separated within a codeword. For brevity, the interleaver and de-interleaver are not shown explicitly in Figure 5-31.

At the receiver, a two-component iterative soft-decision decoder is used. Tentative decisions of the bits expressed as log-likelihood-ratios (LLR) are passed iteratively between a lattice-aware detector and an LDPC decoder until convergence or until a maximum number of iterations is reached. Afterward, the decisions are finalized by comparing the LLRs to a threshold, typically zero. The details of this decoder will be described shortly.

The motivation for the two-component and iterative structure is that the two components each handles a particular aspect of the decoding. The lattice-aware detector deals with the channel distortion and the lattice structure of the tilted-QAM code, and knows what symbols are close to the received point; while the LDPC decoder focus on the redundancy, and knows which bit strings are valid codewords. For optimal decoding, both aspects should be considered simultaneously, i.e., we want a valid codeword that is the closest to the received point. However, directly solving for it would have extremely high complexity. Instead, we iterate between the two components, allow them to exchange information and come to a joint conclusion as their decisions converge.

Next, we describe the iterative soft-decision decoding algorithm in detail.

## 5.6.2 Iterative Soft-Decision Decoder

The detail of the iterative soft-decision decoder is shown in Figure 5-32. The log-likelihood-ratio scores passed between the LDPC decoder and the lattice-aware detector are labeled. These forms of LLR are common to many soft-decision systems.

Let us first look at the LDPC decoder. We do not say much about it here except that the input bit-wise LLR scores it requires are

$$\log \frac{P[\mathbf{Y}|b=1]}{P[\mathbf{Y}|b=0]}, \quad (5.43)$$

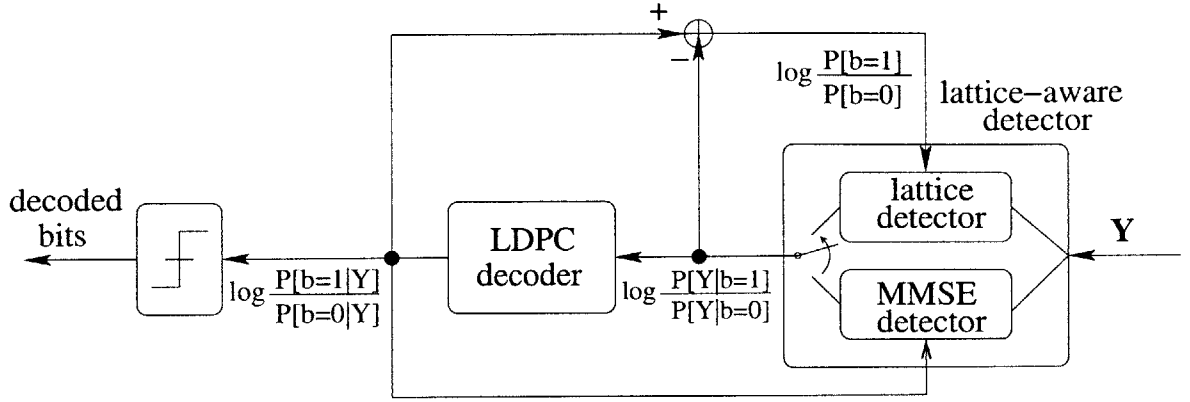


Figure 5-32: Passing of bit-wise LLR scores between an LDPC decoder and a lattice-aware detector unit consisting of a lattice detector and an MMSE detector.

and its output LLR scores have the form

$$\log \frac{P[b=1|Y]}{P[b=0|Y]} = \log \frac{P[b=1, Y]}{P[b=0, Y]} = \log \frac{P[Y|b=1]}{P[Y|b=0]} + \log \frac{P[b=1]}{P[b=0]}. \quad (5.44)$$

More details on the LDPC can be found in [10].

The lattice-aware detector we design has two components as shown in Figure 5-32. The lattice detector tries to treat the discrete constellation exactly. However, due to computationally constraint, approximations must be made. In certain cases where the approximations can not be easily computed, we use the MMSE detector, where the constellation is simply treated as continuous Gaussian with the right means and variances.

The lattice detector takes as input the LLR score,

$$\log \frac{P[b=1]}{P[b=0]} = \log \frac{P[b=1|Y]}{P[b=0|Y]} - \log \frac{P[Y|b=1]}{P[Y|b=0]},$$

which is initialized to all zeros during the first iteration.

We now describe how it computes the LLR score  $\log \frac{P[Y|b=1]}{P[Y|b=0]}$  from the input  $\log \frac{P[b=1]}{P[b=0]}$ , or equivalently,  $P[b=1]$  and  $P[b=0]$ .

For each  $2 \times 2$  block, there are  $M^8$  constellation points labeled by  $8 \log_2 M$  bits, where  $M$  is the constellation size per dimension. Let us use  $\mathbf{b}$  to indicate such a bit

string, and use  $b_i$  for the  $i$ th bit, then we have

$$\log \frac{P[\mathbf{Y}|b_i=1]}{P[\mathbf{Y}|b_i=0]} = \log \frac{P[b_i=1, \mathbf{Y}]}{P[b_i=0, \mathbf{Y}]} - \log \frac{P[b_i=1]}{P[b_i=0]} \quad (5.45)$$

$$= \log \frac{\sum_{\{\mathbf{b}|b_i=1\}} P[\mathbf{Y}|\mathbf{b}] \cdot P[\mathbf{b}]}{\sum_{\{\mathbf{b}|b_i=0\}} P[\mathbf{Y}|\mathbf{b}] \cdot P[\mathbf{b}]} - \log \frac{P[b_i=1]}{P[b_i=0]}. \quad (5.46)$$

To compute  $P[\mathbf{b}]$ , the bits are treated as if they were independent, i.e.,

$$P[\mathbf{b}] = \prod_{j=1}^{8 \log_2 M} P[b_j].$$

This is because the redundancy is ignored by the lattice detector and is only handled by the LDPC decoder.

The conditional probability  $P[\mathbf{Y}|\mathbf{b}]$  is

$$P[\mathbf{Y}|\mathbf{b}] \propto \exp \left\{ -\frac{\|\mathbf{Y} - \mathbf{H}\mathbf{X}(\mathbf{b})\|^2}{2\sigma_w^2} \right\},$$

where  $\mathbf{X}(\mathbf{b})$  is the transmit constellation point corresponding to the bit string  $\mathbf{b}$ . This is a result of the additive white Gaussian noise assumption.

To compute  $\log \frac{P[\mathbf{Y}|b_i=1]}{P[\mathbf{Y}|b_i=0]}$  exactly according to (5.46), we need to sum over all  $M^8$  constellation points. However, this is obviously too computationally intensive. Instead, we approximate it by listing a small number of points with the highest values of  $P[\mathbf{Y}|\mathbf{b}]P[\mathbf{b}]$  and only summing over them. Similar listing technique was used by Hochwald and ten Brink in [14].

To determine the number of points to list, we must find a balance between complexity and performance. We should list as few as possible to keep the complexity low, and list as many as possible so that the approximation is good. In our simulations, we choose to use  $40 \sim 120$  points, which has acceptable complexity, and seem to yield reasonably good performance.

Listing the set of constellation points with the highest values of  $P[\mathbf{Y}|\mathbf{b}]P[\mathbf{b}]$  is done using a specially modified sphere decoder. We will not discuss sphere decoder in detail here except that it locates the closest point in a high dimensional lattice to

a given point. We modify the sphere decoder to list all the points within a certain radius. We also modified it to consider  $2\sigma_w^2 \log(P[\mathbf{b}]) + \|\mathbf{Y} - \mathbf{H}\mathbf{X}(\mathbf{b})\|^2$  as the effective distance measure, so that the likelihood scores are included.

Once we have the list of points with the highest values of  $P[\mathbf{Y}|\mathbf{b}]P[\mathbf{b}]$ , we can then perform the summation in (5.46) over this set for each bit  $b_i$ . From this, the lattice detector can compute its output LLR scores.

The LLR scores computed using the partial sum approximation are good in many cases; however, for some bits, the points listed might all have that bit being 1 or all 0. When this happens, one of the summations in (5.46) would be empty, and no meaningful LLR score can be computed. This would typically happen for MSBs, since points with different MSBs are often far away. For these bits, we use the MMSE detector.

We perform MMSE detection for each symbol in  $\mathbf{S}$  treating all others as continuous interference. Once we obtain the MMSE estimate, we can consider the equivalent channel,  $\hat{s} = s + e$ , and compute LLR scores for each bit in the symbol using the same method as in the D-BLAST-MMSE case (see Figure. 5-10).

As iterations go by, we can use existing soft estimates of the bits in the form of  $\log \frac{P[b=1|\mathbf{Y}]}{P[b=0|\mathbf{Y}]}$  to help reduce the interference. We estimate the mean of the interference and cancel that out. Eventually, if all the bits are known exactly, then there would be no interference left. We choose to use  $\log \frac{P[b=1|\mathbf{Y}]}{P[b=0|\mathbf{Y}]}$  instead of  $\log \frac{P[b=1]}{P[b=0]}$  because it carries more information about the bits and seem to yield better performance empirically.

Combining the lattice detector and the MMSE detector, LLR scores for all bits can be computed.

Note that the main difference between the lattice detector and the MMSE detector is that the former treats the interference as discrete while the latter treats it as continuous. Given a certain received signal point, it is important to treat the nearby points as discrete to have accurate measures of the distances to them, which is done by the lattice detector. For points far away however, the discreteness matters less, allowing us to use the MMSE detector.



### 5.6.3 Simulation Results

Using the system setup and the iterative soft-decision decoder described, we perform three sets of simulations at two different rates,  $R = 6$  and  $8$  b/s/Hz, which are the same rates used in the D-BLAST-MMSE simulations. For the  $R = 6$  b/s/Hz case, we experiment with 16-QAM and 64-QAM constellations to see how the performance is effected by the choice of constellation size. For the  $R = 8$  b/s/Hz case, only 64-QAM constellation is used. At the iterative decoder, a maximum of five iterations are allowed, which appears to be sufficient.

The resulting block error rate curves are plotted in Figure 5-33. The left figure is for  $R = 6$  b/s/Hz, and the right one is for  $R = 8$  b/s/Hz. In each figure, the left most thin line represent the ultimate performance limit associated with channel outage probability, the thick solid lines are for the tilted-QAM-LDPC concatenated system we are evaluating, and the dashed lines are for the D-BLAST-MMSE case (see section 5.3.5), drawn here for comparison.

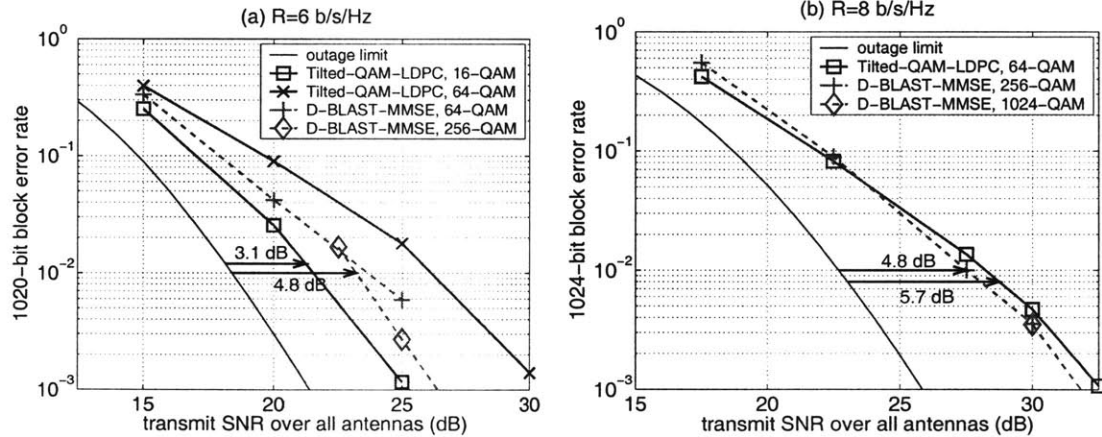


Figure 5-33: Block error rates achieved by tilted-QAM-LDPC concatenated systems (thick solid), compared with D-BLAST-MMSE (dashed), and the ultimate outage probability limit (thin solid), at  $R = 6$  and  $R = 8$  b/s/Hz, using two-transmit two-receive antenna systems.

We first look at the gaps to the ultimate performance limits achieved, as labeled in Figure 5-33. At  $10^{-2}$  target block error rate, tilted-QAM-LDPC can reach 3.1 dB

from capacity in the  $R = 6$  b/s/Hz case. Compared to D-BLAST-MMSE, which is 4.8 dB away, a small improvement in performance is achieved. This is because D-BLAST-MMSE treats the interference between the symbols as Gaussian, when it is really discrete. (This problem was discussed earlier in section 5.3.3.) Tilted-QAM-LDPC, on the other hand, avoids this problem using the lattice-aware detector.

In the  $R = 8$  b/s/Hz case, tilted-QAM-LDPC only reaches 5.7 dB from capacity, and is about 1 dB worse than D-BLAST-MMSE, which is still 4.8 dB away. The reason for this degradation in performance is that, as rate increases, it becomes more difficult to perform the soft-decision decoding. In particular, the lattice detector only considers a small number of neighbors instead of all the constellation points to reduce complexity. The larger the constellation is, the worse the approximation.

We note that in all the experiments performed, no constellation shaping is used. Therefore, all the gaps to capacity quoted include a portion due to the lack of shaping gain. With simple shaping techniques, the resulting gap could be smaller by about 1 dB.

Next, let us turn our attention to the constellation size issue. Compare the two tilted-QAM-LDPC results (thick solid lines) in the  $R = 6$  b/s/Hz case, it is clear that 16-QAM constellation does better than 64-QAM. This is because when the constellation is unnecessarily dense, the points become too close compared to the noise level. When this happens, noise could carry the original signal to many constellation points away, and the transmitted point would not be among the nearby neighbors of the received point, and would not be considered by the lattice detector.

Another constellation size issue worth noting is that tilted-QAM-LDPC uses much smaller constellations compared to D-BLAST-MMSE, while achieving similar performance. Smaller constellations are preferred for ease of implementation. For  $R = 6$  b/s/Hz, tilted-QAM-LDPC uses 16-QAM constellation, while D-BLAST-MMSE uses 256-QAM. This is due to the finite constellation problem of D-BLAST we discussed in section 5.3.3. As tilted-QAM-LDPC demonstrates, this problem is avoidable.

## 5.7 Summary

In this chapter, we study building practical system using long error correction codes in the multiple antenna communication scenario. We combine powerful coding techniques developed for scalar AWGN channels with multiple antenna coding techniques, in particular, OSTBC, various versions of BLAST, and tilted-QAM codes.

We examine two approaches. One is to transform a multiple antenna channel to multiple single antenna channels. This leads to application of error correction codes in a natural way. This is the approach taken by various versions of BLAST with successive cancellation decoding. The other approach is to use a concatenated coding scheme where codes specific to the multiple antenna channel, such as tilted-QAM codes, are used as inner codes, and hard or soft-decision ECC are used outer codes to further enhance the performance of the overall system. A system combining OSTBC and ECC can be interpreted both ways.

We show that OSTBC enhanced with capacity achieving ECC is near optimal in the low SNR regime. It is less than 1 dB from optimal for SNR below 10 dB and rate below 2 b/s/Hz and less than 3 dB from optimal for SNR below 20 dB and rate below 6 b/s/Hz. This suggests that although OSTBC loses multiplexing gain due to repetition, it is still a very attractive method in the low SNR regime. It also has very simple implementation and low decoding complexity. However, OSTBC becomes increasingly sub-optimal in the high SNR regime. Also, there is no good higher-dimensional OSTBC codes for systems with more than two antennas.

While OSTBC can be used for systems operating in low to moderate SNR regimes, coding schemes suitable for high SNR still needs further study. Current designs like diagonal-BLAST is theoretically optimal, but its non-block form leads to practical problems like error propagation. Other block form variations like V-BLAST and X-BLAST would be optimal or close to being optimal if joint decoding can be done. However, there is no efficient joint decoding scheme at this time.

We experiment with combining tilted-QAM code with hard and soft decision error correction codes. The complexity of the joint decoding involved is feasible for off-line

simulations but beyond reach for current real time systems, especially in the case of soft decoding. At the target error rate of  $10^{-2}$ , the resulting gap to capacity is about 5.2 dB for hard decision systems. For soft-decision ones, the gap is 3.1 dB for  $R = 6$  b/s/Hz and 5.7 dB for  $R = 8$  b/s/Hz. In comparison, D-BLAST-MMSE is 4.8 dB from capacity at those rates.

From our study, it is evident that more work is needed to design good systems for multiple antenna communication in the high SNR regime and using more antennas, and new design criteria might be needed.

## Chapter 6

# Non-Coherent Communications

In the earlier chapters, we have assumed perfect channel knowledge at the receiver. However, in many practical applications, this assumption can not be satisfied exactly. For example, the channel might be varying too fast to be tracked accurately. This scenario is appearing more and more often as wireless devices are now operating at higher and higher carrier frequencies while the lower frequency spectrum becomes quickly filled. Even when the channel is varying more slowly, we might only be able to track it to some degree, instead of having perfect channel knowledge.

In this chapter, we investigate the case of non-coherent communication, where the channel knowledge at the receiver is absent or imperfect. Again, we assume that the transmitter has no channel knowledge. We continue to model the channel as  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , where the  $N_r \times N_t$ ,  $N_r \geq N_t$ , matrix  $\mathbf{H}$  represents the flat, Rayleigh, and block fading channel, and  $\mathbf{W}$  is the additive white Gaussian noise. Entries of  $\mathbf{H}$  and  $\mathbf{W}$  are independent with identical distribution  $\mathcal{CN}(0, 1)$ . The average energy of each entry of  $\mathbf{X}$  is  $\rho$ , while that of each column of  $\mathbf{X}$  is  $\text{SNR} = N_t\rho$ .

There are still many unanswered question for this non-coherent communication problem. Neither exact capacity formulas nor efficient encoding and decoding algorithms exist. Only some aspects of the achievable capacity are known and there exist some inefficient or sub-optimal coding algorithms.

We first review some existing work on theoretical results, design rules, as well as some specific designs. Then, we propose a geometric approach that links the non-

coherent signal design problem to that of the coherent case with training, where some predetermined, non-information bearing signal is sent to probe the channel. We argue that the training approach is not too far from being optimal. We also look at what decoding performance can be achieved using the channel estimates obtained through training.

## 6.1 Theoretical Background

### 6.1.1 Capacity

The capacity of a non-coherent system is clearly upper bounded by the capacity of a system where the channel coefficients are perfectly known by the receiver, which is [36]

$$C_u = \log_2 \left( \det \left( \mathbf{I}_{N_r} + \rho \mathbf{H} \mathbf{H}^\dagger \right) \right), \quad (6.1)$$

where  $\rho = \text{SNR}/N_t$  is the average transmit SNR per antenna.

This capacity grows linearly with  $\log_2 \rho$  in the limit of high SNR. Specifically, for every 3 dB increase in SNR, the capacity grows by  $K$  b/s/Hz, where  $K = \min(N_t, N_r)$ . For non-coherent systems, it turns out that the growth rate is slightly modified.

Zheng and Tse studied the capacity of non-coherent systems in the limit of high SNR, assuming block fading model in [40]. They showed that non-coherent capacity also grows linearly in the limit of high SNR. The difference is that for every 3 dB increase in SNR, the capacity grows by  $K(1 - K/T)$  b/s/Hz, where  $K = \min(N_t, N_r, \lfloor T/2 \rfloor)$  and  $T$  is the coherence time of the channel, which is the time that the channel remains constant before independently changing to other values.

While the full proof is more involved, the intuition behind the capacity growth rate result is a dimensionality count. For simplicity, let us look at the symmetric case,  $K = N_t = N_r \leq T/2$ . At the transmitter, with  $K$  transmit antennas and a block of  $T$  times, we have  $KT$  degrees of freedom. At the receiver, we need to

solve for the channel, which has  $N_t N_r = K^2$  degrees of freedom. So we are left with  $KT - K^2 = K(T - K)$  degrees of freedom for transmitting information. Normalize this by time, we have  $K(1 - K/T)$ .

Furthermore, [42] shows that a non-coherent system with  $N_t$  transmit antennas <sup>1</sup>,  $N_r$  receive antennas, and block length  $T$ , has similar diversity-multiplexing tradeoff as a coherent system with the same number of antennas and block length  $T - N_t$ . The difference is that the tradeoff curve achieved is scaled in the direction of the multiplexing gain, the  $r$  axis, by a factor of  $(1 - N_t/T)$ . This tradeoff can be achieved with training, where a pilot signal of duration  $N_t$  is first sent to allow the receiver to learn the channel and then the system is treated as coherent. The training phase is what causes the reduction of the factor of  $(1 - N_t/T)$  in multiplexing gain. In the limit of  $T \rightarrow \infty$ , or slow fading,  $(1 - N_t/T)$  approaches 1. The non-coherent capacity meets the perfect-knowledge upper bound as expected. Since for large coherence time, we can spend some time to learn the channel first with negligible cost in rate.

One crucial assumption made in [42] is that the channel stays constant exactly during the coherence time of the channel, so that we can learn the channel to any precision desired and then treat the channel as known. However, this block fading model is not accurate. In reality, channel varies continuously, so that future channel coefficients can be estimated, but not perfectly. In [16], Lapidot and Moser modeled the channel as jointly stationary and ergodic stochastic processes. They concluded that at high SNR, capacity grows double-logarithmically in SNR and not logarithmically as in the block fading case. They suggested in [17] that this double-logarithmic behavior is dominant when the transmission rate significantly exceeds a fading number, which is typically increased by using multiple antennas. At this point, let us assume that the regime we are interested in is below this fading number, so that the block fading model is still reasonably good.

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<sup>1</sup>Assume all  $N_t$  transmit antennas are used in this statement. Sometimes, it is preferable not to use all antennas available.

### 6.1.2 Capacity Achieving Distribution

In order to design good coding schemes, we first look at the capacity achieving distribution, which was studied in [24]. Given a particular distribution on the transmitted signal matrix  $\mathbf{X}$ , the mutual information between the input and output of the channel is defined as,

$$I(\mathbf{Y}; \mathbf{X}) = E_{\mathbf{Y}, \mathbf{X}} \left[ \log \left( \frac{p(\mathbf{Y}|\mathbf{X})}{E_{\mathbf{Y}}[p(\mathbf{Y}|\mathbf{X})]} \right) \right]. \quad (6.2)$$

The capacity achieving distribution is the distribution  $p(\mathbf{X})$  that maximizes the mutual information to achieve capacity,

$$C = \sup_{p(\mathbf{X})} I(\mathbf{Y}; \mathbf{X}). \quad (6.3)$$

The mutual information  $I(\mathbf{Y}; \mathbf{X})$  is a function of both  $p(\mathbf{Y}|\mathbf{X})$  and  $p(\mathbf{X})$ . Therefore, the distribution  $p(\mathbf{X})$  that maximizes  $I(\mathbf{Y}; \mathbf{X})$  depends on  $p(\mathbf{Y}|\mathbf{X})$ , which we now look at. In the case of  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$ , where the entries of  $\mathbf{H}$  and  $\mathbf{W}$  are IID  $\mathcal{CN}(0, 1)$  random variables, and  $\mathbf{X}$  is  $N_t \times T$ , we have

$$p(\mathbf{Y}|\mathbf{X}) = \frac{\exp \left( -\text{tr} \left( (\mathbf{I}_T + \mathbf{X}^\dagger \mathbf{X})^{-1} \mathbf{Y}^\dagger \mathbf{Y} \right) \right)}{\pi^{TN_r} (\det(\mathbf{I}_T + \mathbf{X}^\dagger \mathbf{X}))^{N_r}}. \quad (6.4)$$

We see that the effect of  $\mathbf{X}$  appears only through  $\mathbf{X}^\dagger \mathbf{X}$ , so we can multiply  $\mathbf{X}$  by any unitary matrix on the left without changing  $p(\mathbf{Y}|\mathbf{X})$ . We can perform singular value decomposition on  $\mathbf{X}$  and factor it into  $\mathbf{X} = \Psi \Lambda \Phi^\dagger$ , where  $\Psi$  and  $\Phi$  are unitary and  $\Lambda$  is diagonal. Since we can remove  $\Psi$  by multiplying  $\Psi^\dagger$  on the left without changing  $p(\mathbf{Y}|\mathbf{X})$ , we can limit  $\mathbf{X}$  to take the form of just  $\mathbf{X} = \Lambda \Phi^\dagger$  with no loss of generality. With some additional rotational symmetry argument, it turns out that the capacity distribution is given by the following lemma [24].

**Lemma 6.1** *The signal matrix that achieves capacity can always be factored as  $\mathbf{X} = \Lambda \Phi^\dagger$ , where  $\Phi$  is an  $T \times N_t$  isotropically distributed unitary matrix, and  $\Lambda$  is an independent  $N_t \times N_t$  real, non-negative, diagonal matrix.*



An isotropically distributed unit vector is a unit length vector that is equally likely to point in any direction. An isotropically distributed unitary matrix is a matrix whose columns are isotropically distributed unit vectors that also satisfy the orthogonality constraint, i.e., the second column is a vector that is equally likely to point in any direction that is orthogonal to the first, and so on. One property of isotropically distributed unitary matrix is rotational symmetry, i.e.,  $p(\Phi) = p(\Theta\Phi)$ , for any unitary matrix  $\Theta$ .

The capacity achieving distribution for the diagonal matrix  $\Lambda$  is not given in [24] except in the limiting case. For fixed  $N_t$ , as  $T \rightarrow \infty$ , the optimal distribution of  $\Lambda$  converges to  $\sqrt{\rho T} \mathbf{I}_{N_t}$  in probability. This means that when  $T$  is large enough, we can pick all diagonal entries of  $\Lambda$  to be  $\sqrt{\rho T}$ , so that,  $\mathbf{X} = \sqrt{\rho T} \Phi^\dagger$ . For the case of  $N_t = N_r = 1$ , this approximation is shown to be good when  $T > 12$  at SNR = 0 dB,  $T > 4$  at SNR = 6 dB, and  $T > 3$  at SNR = 12 dB [24]. We see that the lengths required are not very large and can be satisfied easily in practice.

It seems intuitive that the capacity achieving distribution should be isotropic when the channel is Rayleigh fading and the transmitter has absolutely no knowledge of the realized channel. However, if the transmitter were to have some side information about the realized channel through feedback or if the channel were not Rayleigh fading, then the symmetry would be broken and the capacity achieving distribution would be different. Some of these issues are investigated in [37].

In comparison, the capacity achieving distribution for the AWGN channel is simply Gaussian with zero mean and variance matching the signal power constraint. The capacity achieving distribution of the non-coherent multiple antenna channel is much more complicated, and the coding problem is more difficult as well.

## 6.2 Non-Coherent Communication Signal Design

By looking at the capacity achieving distribution, we have established that when  $T$  is sufficiently large, we can pick the transmitted signal matrix for message  $l$ ,  $l \in \{1, \dots, L\}$ , to be  $\mathbf{X} = \sqrt{\rho T} \Phi_l^\dagger$ , where  $\Phi_l$  is an  $T \times N_t$  unitary matrix and  $L$

is the size of the signal set. This scheme is called *Unitary Space-Time Modulation*. Geometrically, unitary matrices, or the subspaces they span, are used to represent messages. In comparison, points are used for AWGN channels.

### 6.2.1 Design Criterion

In this study, the  $N_t \times T$  transmitted signal matrix  $\Phi$  is considered as one codeword, and we are interested in detecting which one is transmitted from the received signal  $\mathbf{Y} = \sqrt{\rho T} \mathbf{H} \Phi_l^\dagger + \mathbf{W}$ .

The maximum-likelihood detector for this signal set is  $\Phi_{\text{ML}} = \arg \max_{\Phi_l} p(\mathbf{Y}|\Phi_l)$ . After some mathematical manipulation, it turns out to be [12]

$$\Phi_{\text{ML}} = \arg \max_{\Phi_l \in \{\Phi_1, \dots, \Phi_L\}} \text{tr}(\mathbf{Y} \Phi_l^\dagger \Phi_l \mathbf{Y}^\dagger). \quad (6.5)$$

This ML detector maximizes the energy contained in the product  $\mathbf{Y} \Phi_l^\dagger$ , similar to match filtering. With this detector, the pair-wise error probability, the probability of confusing  $\Phi_1$  and  $\Phi_2$  while ignoring other ones, has a Chernoff upper bound of

$$P[\Phi_1 \rightarrow \Phi_2] \leq \frac{1}{2} \prod_{n=1}^{N_t} \left[ \frac{1}{1 + \frac{(\rho T / N_t)^2 (1 - d_n^2)}{4(1 + \rho T / N_t)}} \right]^{N_r}, \quad (6.6)$$

where  $1 \geq d_1 \geq \dots \geq d_{N_t} \geq 0$  are the singular values of the  $N_t \times N_t$  correlation matrix  $\Phi_2^\dagger \Phi_1$ .

From the above equation, we see that the probability of error decreases with decreasing  $d_n$ . Geometrically, these  $d_n$ 's correspond to the cosine's of a set of *principle angles* defined between the two subspaces spanned by  $\Phi_1$  and  $\Phi_2$ . Therefore, to design a good signal set that has low probability of error, we need to find a set of unitary matrices, such that the subspaces they span are well separated in terms of the angles between them.

### 6.2.2 Existing Schemes

Next, we review some existing signal design schemes.

**Iterative Search Method** One iterative method of searching for a good deterministic signal set was described in [12]. The basic idea is to start from an initial signal set and improve it iteratively. We first compute all pairs of correlation matrices  $\Phi_i^\dagger \Phi_j$ , and then identify the worst pair and try to “move them apart”. This procedure is repeated until improvement diminishes. This method is computationally intensive, especially when the signal set is large. For a signal set of size  $L$ , there are  $L(L-1)/2$  pairs of correlations to compute. Therefore, this method is only applicable in low dimensional and low rate cases.

**Systematic Design of Unitary Space-Time Signals** A systematic method of designing good unitary space-time signal sets was developed in [13]. The design is not optimal, but the design complexity is low. The idea is to design one unitary matrix  $\Theta$  and use it to generate all signal matrices in the set. More specifically, the design proposed is

$$\Phi_l = \Theta^{l-1} \Phi_1, \text{ for } l = 1, \dots, L, \quad (6.7)$$

where  $\Theta$  is a  $T \times T$  unitary matrix such that  $\Theta^L = \mathbf{I}_T$ , and  $\Phi_1$  is a  $T \times N_t$  unitary matrix. The advantage of this design is that, for any two  $\Phi_i$  and  $\Phi_j$ , the correlation is  $\Phi_1^\dagger \Theta^{((j-i) \bmod L)} \Phi_1$ . Every signal matrix forms the same set of correlation matrices with all the other signal matrices. Now there are only  $L-1$  correlation matrices to check, instead of  $L(L-1)/2$ . To further reduce the design complexity,  $\Theta$  is restricted to be diagonal with entries  $\Theta_{tt} = e^{j(2\pi/L)u_t}$  for  $t = 1, \dots, T$ , where  $u_t$  are integers satisfying  $0 \leq u_1 \leq \dots \leq u_T \leq L-1$ . Now we only need to search over a finite set of integers to find the best  $\Theta$ .

**Unitary Space-Time Autocoding Constellations** Another structured way of designing unitary space-time signal sets was proposed in [23]. A signal set of size

$L = 2^{RT}$  takes the form of

$$\Phi_{l_1 l_2 \dots l_{RT}} = \Omega_1^{l_1} \Omega_2^{l_2} \dots \Omega_{RT}^{l_{RT}} \Phi_{00 \dots 0}, \quad l_1, l_2, \dots, l_{RT} \in \{0, 1\}, \quad (6.8)$$

where  $\Phi_{00 \dots 0}$  is  $T \times N_t$  and  $\Omega_1, \dots, \Omega_{RT}$  are  $T \times T$  independent isotropically distributed unitary matrices.<sup>2</sup> A desirable property of this design is that, statistically, every signal set has signal matrices that are pair-wise independent and marginally isotropically random. This statistics is capacity achieving.

**Differential Coding** Differential coding is well studied for the non-coherent single antenna system, and was extended to multiple antenna systems by various groups [11, 33]. In this case,  $N_t \times N_t$  matrices are used to represent messages. The idea is to utilize the fact that channel does not change much between these short blocks, and transmit

$$\mathbf{X}_\tau = \mathbf{X}_{\tau-1} \Phi_l^\dagger \quad (6.9)$$

for message  $l$  during block  $\tau$ . This differential scheme leads to

$$\mathbf{Y}_\tau = \mathbf{H}(\mathbf{X}_{\tau-1} \Phi_l^\dagger) + \mathbf{W}_\tau = \mathbf{Y}_{\tau-1} \Phi_l^\dagger + (\mathbf{W}_\tau - \mathbf{W}_{\tau-1} \Phi_l^\dagger). \quad (6.10)$$

The effective channel is  $\mathbf{Y}_{\tau-1}$ , which is perfectly known at the receiver. The detection problem becomes a coherent one. However, the down-side is that the noise from the previous block,  $\mathbf{W}_{\tau-1}$ , propagates through, so that the effective noise power is doubled. Note that, for differential coding, the block fading model is not used.

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<sup>2</sup>Detection for this signal set is like “trying to pick a combination lock”. There is no efficient algorithm but exhaustive search.

## 6.3 Geometric Approach

The criterion for designing good codes for non-coherent communication is to have a set of unitary matrices such that the subspaces they span are well separated in terms of the angles between them. Most of the existing schemes design unitary matrices algebraically. In this section, we introduce a geometric approach. This is based on work by Conway and Sloane on packing low dimensional subspaces in high dimensions [5].

### 6.3.1 Projection Matrices

Subspaces have a one-to-one relationship with projection matrices corresponding to projection onto that subspace. A subspace spanned by a  $T \times N_t$  unitary matrix  $\Phi$  corresponds to a  $T \times T$  projection matrix  $\mathbf{P} = \Phi\Phi^\dagger$  with rank  $N_t$ . The task of designing good signal sets can be turned into choosing a set of projection matrices that are well separated in terms of a Euclidean distance, which is a metric that we are much more familiar with and can visualize easily.

The Euclidean distance between two matrices is defined as the  $L_2$  norm of the difference matrices. Let  $\mathbf{P}_1 - \mathbf{P}_2 = \Delta$ , and let  $\delta_{ij}$  be the entries of  $\Delta$ ,

$$\|\mathbf{P}_1 - \mathbf{P}_2\|^2 = \|\Delta\|^2 = \sum \sum \delta_{ij}^2 = \text{tr}(\Delta^\dagger \Delta) = \text{tr}(\Delta \Delta^\dagger). \quad (6.11)$$

Consider two unitary signal matrices  $\Phi_1$  and  $\Phi_2$ . Their corresponding projection matrices are  $\mathbf{P}_1 = \Phi_1\Phi_1^\dagger$  and  $\mathbf{P}_2 = \Phi_2\Phi_2^\dagger$ . Using the projection matrix properties,  $\mathbf{P} = \mathbf{P}^\dagger$ ,  $\mathbf{P} = \mathbf{P}^2$ , and  $\text{tr}(\mathbf{P}) = \text{rank}(\mathbf{P})$ , we have,

$$\begin{aligned} \|\mathbf{P}_1 - \mathbf{P}_2\|^2 &= \text{tr}(\mathbf{P}_1^2) + \text{tr}(\mathbf{P}_2^2) - 2 \cdot \text{tr}(\mathbf{P}_1\mathbf{P}_2) \\ &= N_t + N_t - 2 \cdot \text{tr}(\Phi_1\Phi_1^\dagger\Phi_2\Phi_2^\dagger) \\ &= 2(N_t - \|\Phi_2^\dagger\Phi_1\|^2) \\ &= 2 \sum_{i=1}^{N_t} (1 - d_i^2), \end{aligned} \quad (6.12)$$

where  $\{d_i\}$  is the set of singular values of  $\Phi_2^\dagger \Phi_1$ . Recall that the error probability of the ML detector (6.6) decreases with decreasing  $d_i$ . So maximizing the Euclidean distance between  $\mathbf{P}_1$  and  $\mathbf{P}_2$  can lead to smaller  $d_i$  and lower probability of error.

It happens that the ML detector also has a geometric interpretation of minimizing Euclidean distances between matrices.

$$\Phi_{\text{ML}} = \arg \max_{\Phi_l} (\mathbf{Y} \Phi_l^\dagger \Phi_l \mathbf{Y}^\dagger) = \arg \min_{\Phi_l} (\|\mathbf{Y}^\dagger \mathbf{Y} - \Phi_l^\dagger \Phi_l\|^2). \quad (6.13)$$

Note that  $\mathbf{Y}^\dagger \mathbf{Y}$  is not a projection matrix, just a Hermitian matrix.

### 6.3.2 Embedding on Spheres

Let us now look at some properties that can help us visualize the geometry of all projection matrices. Let us simplify the problem by treating all matrices as real for the purpose of developing geometric intuitions.

First of all, all real projection matrices are symmetric, and the set of  $T \times T$  symmetric matrices forms an Euclidean space. Summing and scaling symmetric matrices result in symmetric matrices and we can use the  $L_2$  norm between matrices (6.11) as the distance metric in this space.

This space of symmetric matrices is the space we operate in. We can show that all projection matrices are embedded on a sphere centered at  $\mathbf{I}_T/2$  with radius  $\sqrt{T}/2$ .

$$\begin{aligned} \left\| \mathbf{P} - \frac{\mathbf{I}_T}{2} \right\|^2 &= \text{tr} \left( \mathbf{P}^2 - \mathbf{P} + \frac{\mathbf{I}_T}{4} \right) \\ &= \text{tr} \left( \frac{\mathbf{I}_T}{4} \right) = \frac{T}{4}. \end{aligned} \quad (6.14)$$

All the projection matrices we are interested in for signal design are rank  $N_t$ . We now show using similar techniques that the set of rank  $N_t$  projection matrices are embedded on a lower dimensional sphere centered at  $(N_t/T)\mathbf{I}_T$  with radius

$$\sqrt{N_t(1 - N_t/T)}.$$

$$\begin{aligned} \left\| \mathbf{P} - \frac{N_t}{T} \mathbf{I}_T \right\|^2 &= \text{tr} \left( \mathbf{P}^2 - \frac{2N_t}{T} \mathbf{P} + \frac{N_t^2}{T^2} \mathbf{I}_T \right) \\ &= \left( 1 - \frac{2N_t}{T} \right) N_t + \frac{N_t^2}{T^2} T = N_t \left( 1 - \frac{N_t}{T} \right) \end{aligned} \quad (6.15)$$

Recall that at high SNR, capacity grows by  $K(1 - K/T)$  b/s/Hz for every 3 dB increase in SNR, where  $K = \min(N_t, N_r, \lfloor T/2 \rfloor)$ . We see that when  $K = N_t$ , capacity growth rate is determined by the radius of the sphere. The intuition is that the larger the sphere, the more points we can choose for a given minimum distance criterion, the greater the capacity.

The geometrical properties of projection matrices in (6.14) and (6.15) are summarized in Figure 6-1.

### 6.3.3 Signal Design

We have now rephrased the signal design problem to one of finding well separated points on a sphere. However, doing it systematically is still difficult. Here, we propose to use an approximation of the sphere to allow systematic construction of a signal set (constellation).

The idea is to approximate the sphere with a set of tangential planes as shown in Figure 6-2. The signal design can now be decomposed into two parts. Within each plane, we can design a constellation (small dots) using existing techniques for choosing well separated points in a Euclidean space, for example, a lattice code. We call this the fine constellation. Across planes, we need to choose where the tangent points are (large dots). We call them the coarse constellation. Their design still has the original non-coherent signal design complexity. Note that the more planes there are, the more accurate the approximation of the sphere; however, there are more coarse constellation points to be chosen, which increases design complexity.

Next, we translate the geometric intuition into more precise algebraic expressions.

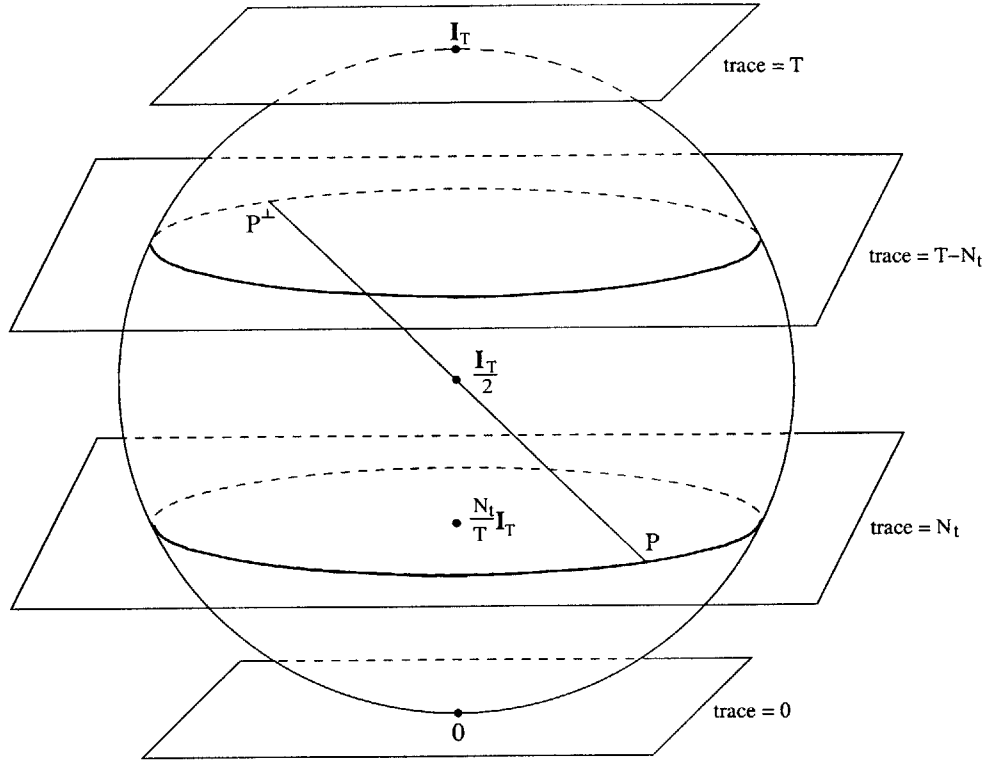


Figure 6-1: In the space of symmetric matrices, all projection matrices (of any rank) are embedded on (the surface of) a sphere centered at  $\mathbf{I}_T/2$  with radius  $\sqrt{T}/2$ . Projection matrices with a particular trace (rank) are embedded on lower dimensional spheres. This figure is from [5].

One projection matrix of rank  $N_t$  is  $\begin{bmatrix} \mathbf{I}_{N_t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ . Without loss of generality, let us use it as one of the coarse constellation points. The points on the plane tangent to the sphere at this point can be described by  $\begin{bmatrix} \mathbf{I}_{N_t} & \Psi^\dagger \\ \Psi & \mathbf{0} \end{bmatrix} \approx \begin{bmatrix} \mathbf{I}_{N_t} \\ \Psi \end{bmatrix} \begin{bmatrix} \mathbf{I}_{N_t} & \Psi^\dagger \end{bmatrix}$ . Note that  $\Psi$  is an  $N_t \times (T - N_t)$  matrix, so the total degrees of freedom normalized by  $T$  is  $N_t(1 - N_t/T)$ . Again, this is the capacity growth rate when  $N_t = \min(N_t, N_r, \lfloor T/2 \rfloor)$ .

All coarse constellation points are rank  $N_t$  projection matrices. They can be written in the form of  $\Omega \begin{bmatrix} \mathbf{I}_{N_t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Omega^\dagger$ , where  $\Omega$  is a  $T \times T$  unitary matrix. Fine constellation

points on planes tangent at those points can be described by  $\Omega \begin{bmatrix} \mathbf{I}_{N_t} & \Psi^\dagger \\ \Psi & \mathbf{0} \end{bmatrix} \Omega^\dagger$ .



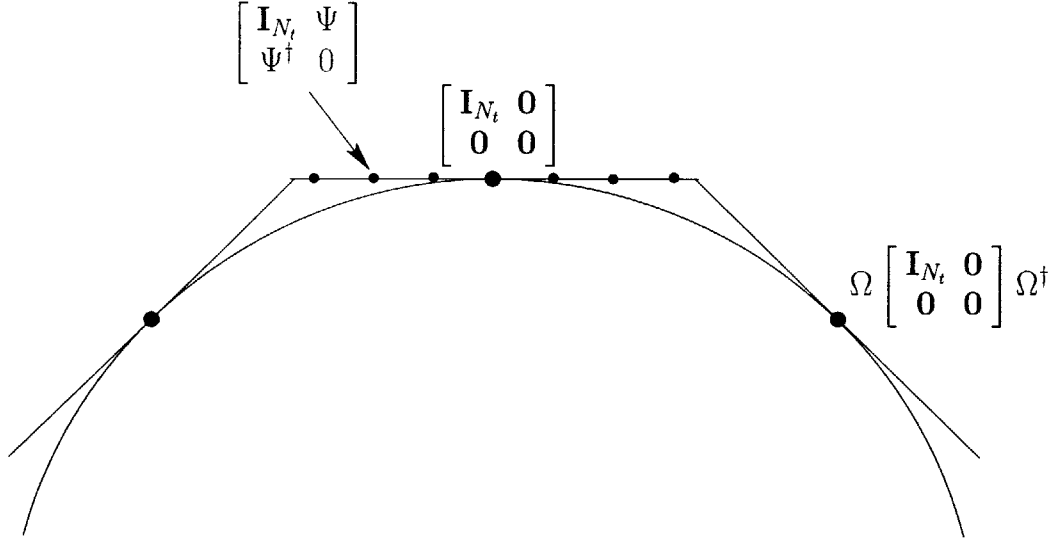


Figure 6-2: Using a “polygon” approximation to design a set of well separated points on a sphere.

Therefore, a signal matrix described by coarse constellation point  $i$  and fine constellation point  $j$  is

$$\Phi_{ij} = \Omega_i \begin{bmatrix} \mathbf{I}_{N_t} \\ \Psi_j \end{bmatrix}. \quad (6.16)$$

We can design the set of coarse constellation points  $\{\Omega_i\}$  and the set of fine constellation points  $\{\Psi_j\}$  separately.

### 6.3.4 Relationship to Training

We can relate the geometric view in Figure 6-2 to the training approach. If we only consider the tangential plane through  $\begin{bmatrix} \mathbf{I}_{N_t} & 0 \\ 0 & 0 \end{bmatrix}$ , the constellation points on it correspond to  $\Phi_j = \begin{bmatrix} \mathbf{I}_{N_t} \\ \Psi_j \end{bmatrix}$ . This can be considered as first sending  $\mathbf{I}_{N_t}$  to allow the receiver to estimate the channel, and then transmit data using  $\Psi_j$ . This is essentially the training approach. If we also consider using multiple tangential planes, then we can potentially convey additional information during the training phase. Therefore,

this geometric coding design can be considered as a modified training approach where the pilot signal can be one of many. This scheme is more complex, but has the potential to increase data rate.

This additional amount of information turns out to be negligible, at least in the high SNR limit. As SNR increases, the fine constellation points can become denser and denser to take advantage of the smaller noise. However, no new tangential planes can be added. Therefore, the amount of information that can be carried during the training phase does not grow with SNR, and becomes negligible in the high SNR limit. In fact, Zheng and Tse showed that training is optimal in terms of diversity-multiplexing tradeoff [42]. This is because multiplexing gain only focuses on growth rate; a constant number of bits does not matter.

In summary, this geometric approach translates the non-coherent signal design problem to one of packing points on a sphere. Further approximation of the sphere using a set of tangential planes leads to a scheme which is a modified training approach where the pilot signal itself can carry some fixed amount of information. This geometric view together with Zheng and Tse's result suggests that training is a reasonably good approach.

## 6.4 Channel Training Approach

In this section, we focus on a training approach for non-coherent communication. In the training scheme we consider, the transmitter first transmit a training signal  $\sqrt{\text{SNR}}\mathbf{I}_{N_t}$  for a period of  $N_t$ , and then spend the rest of coherence time  $T - N_t$  transmitting data as if the channel is perfectly known at the receiver. This is a very simplistic scheme. More optimally, the transmitter should consider the fact that the receiver only has an estimate of the channel after the training phase. But this is more difficult.

One issue we would like to discuss briefly here relates to the time varying nature of the channel and the block fading approximation. If the channel were truly block fading, then the channel we experience during the training and data-transmission

phases would be truly identical. We would be able to first spend some time to obtain a sufficiently good channel estimate and then use it for the rest of the block. However, in reality, the channel varies in time. Even if we had obtained a very good estimate of the channel during the training phase, the true channel would have drifted by the data-transmission phase. As time goes on, the deviation would increase until the next training phase. Médard, Abou-Faycal, and Madhow [25] studied the possibility of transmitting at higher rates during times closer to the pilot signal when the channel estimation error is less and transmitting at lower rates in between when the error is greater.

In this section, let us still assume the block fading model and focus on the simple scheme of training plus coherent communication. We first discuss how the receiver can estimate the channel and what the quality is, and then look at how the performance is affected by imperfect channel knowledge.

### 6.4.1 Quality of Channel Estimation

During the training phase, the transmit signal matrix  $\mathbf{X}$  is the pilot signal  $\mathbf{X} = \sqrt{\text{SNR}}\mathbf{I}_{N_t}$ .<sup>3</sup> The scaling factor  $\sqrt{\text{SNR}}$  is chosen such that the power used during the training phase is the same as the average power used during data-transmission. It is a reasonable thing to do. It also turns out to be very convenient. If the SNR available increases and we want to transmit at a higher rate, we will need to have higher quality channel estimation, and using more energy during training gives us that.

When  $\mathbf{X} = \sqrt{\text{SNR}}\mathbf{I}_{N_t}$ , we have

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W} = \sqrt{\text{SNR}}\mathbf{H} + \mathbf{W}, \quad (6.17)$$

---

<sup>3</sup>Other signals may be used, for example, one with reduced peak power. However, there is no significant difference in this context.

Written in a component-wise form,

$$y_{ij} = \sqrt{\text{SNR}} h_{ij} + w_{ij}. \quad (6.18)$$

We can perform scalar minimum mean square error (MMSE) estimation to estimate  $h_{ij}$ . The reason for choosing the MMSE estimator is so that the resulting estimation error is independent from the estimate, and its variance is minimized.

Using standard MMSE formulation, the resulting estimate is

$$\hat{h}_{ij} = \frac{\sqrt{\text{SNR}}}{\text{SNR} + 1} y_{ij}. \quad (6.19)$$

Let us denote the estimation error with  $\Delta_{\mathbf{H}} = \mathbf{H} - \hat{\mathbf{H}}$  and  $\delta_{h,ij} = h_{ij} - \hat{h}_{ij}$ . It is easy to show that all  $\delta_{h,ij}$  are IID, circularly symmetric, complex Gaussian random variables with density  $\mathcal{CN}(0, 2\sigma_h^2)$ , where

$$2\sigma_h^2 = \frac{1}{\text{SNR} + 1}. \quad (6.20)$$

Note that, as the SNR available to us increases, the variance decreases, and the channel estimation quality improves. Next, let us look at how the quality of the channel estimation affects the system performance.

### 6.4.2 Effect of Imperfect Channel Knowledge

After training, we obtain an estimate of the true channel coefficients. Conditioned on the received signal  $y_{ij}$ , each channel coefficient  $h_{ij}$  is complex Gaussian with density  $\mathcal{CN}\left(\frac{\sqrt{\text{SNR}}}{\text{SNR}+1} y_{ij}, \frac{1}{\text{SNR}+1}\right)$ . Therefore, we effectively have a *Rician channel*. The receiver knows both the mean and the variance of the channel coefficients, while the transmitter only knows the variance but not the mean. This is different from the coherent case, where  $\mathbf{H}$  is deterministic at the receiver and is also different from the non-coherent case, where  $\mathbf{H}$  has (zero mean) Rayleigh density.

Ideally, during the data-transmission phase, the transmitter and receiver should employ a scheme specific for the Rician channel. However, this is beyond the scope of

our study. Instead, we simply use the coherent communication schemes studied earlier and treat the channel estimation error as a form of noise. We discuss its performance and argue that there is effectively no loss in terms of diversity and multiplexing gains.

We can re-write the channel during the data-transmission phase as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W} = \hat{\mathbf{H}}\mathbf{X} + \Delta_{\mathbf{H}}\mathbf{X} + \mathbf{W}. \quad (6.21)$$

In the first term,  $\hat{\mathbf{H}}$  is perfectly known to the receiver. This is similar to coherent detection. The second term  $\Delta_{\mathbf{H}}\mathbf{X}$  can be treated as a new additive noise. Entries of  $\Delta_{\mathbf{H}}$  have variance  $\frac{1}{\text{SNR}+1} \approx \frac{1}{\text{SNR}}$  and  $\mathbf{X}$  has energy on the order of SNR. Consequently,  $\Delta_{\mathbf{H}}\mathbf{X}$  has energy on the order of 1, which is also the noise variance. This means that the new noise  $\Delta_{\mathbf{H}}\mathbf{X}$  and the original AWGN  $\mathbf{W}$  have energy on the same order. Therefore, by doing simple coherent communication treating  $\hat{\mathbf{H}}$  as the true  $\mathbf{H}$ , we only increase the amount of noise by a constant factor. Effectively, this cost only appears as a constant dB loss in SNR, which means that the diversity and multiplexing gains achieved are not affected.

The question of how good channel estimation needs to be was also studied by Lapidot and Shamai in [18]. They suggested that the channel estimation error should be small compared to  $\frac{1}{\text{SNR}}$  to avoid significant performance degradation.

It is worth noting that the new noise  $\Delta_{\mathbf{H}}\mathbf{X}$  is not quite the same as the original AWGN  $\mathbf{W}$ . First of all, for a given  $\mathbf{X}$ , the variance of  $\Delta_{\mathbf{H}}\mathbf{X}$  is a function of  $\mathbf{X}$ . Although the variance is on the order of 1, it does fluctuate with  $\mathbf{X}$ . Also, averaging over  $\mathbf{X}$ , the distribution of  $\Delta_{\mathbf{H}}\mathbf{X}$  is not really Gaussian. It is the average of many zero mean Gaussian distributions. This being said, it is plausible that treating it as Gaussian probably does not influence the performance much.

There is another important difference between the noise terms. After the initial training,  $\Delta_{\mathbf{H}}$  is fixed within one block. If it happens to be large, then we are stuck with a large noise for an entire block of  $T$ , even though  $\Delta_{\mathbf{H}}\mathbf{X}$  has energy of order 1 on average. On the other hand, all entries of  $\mathbf{W}$  are independent. The consequence of this difference is that, while we can use coding to average large and small entries of

$\mathbf{W}$  within one block, we are not able to do that for  $\Delta_{\mathbf{H}}$ . However, since short codes can achieve the same tradeoff as long ones, we expect the inability to do coding to not affect the diversity-multiplexing tradeoff.

Earlier when we argued for  $\Delta_{\mathbf{H}}\mathbf{X}$  and  $\mathbf{W}$  having energy on the same order, we needed the variance of  $\Delta_{\mathbf{H}}$  to be on the order of  $\frac{1}{\text{SNR}}$ . This is true if we have true block fading and the training signal has energy on the order of SNR. If the channel were slowly varying, then a certain time after the training phase, the channel would have drifted by a certain amount independent of SNR. This would contribute to a component of the estimation error that is not order  $\frac{1}{\text{SNR}}$ . When this happens,  $\Delta_{\mathbf{H}}\mathbf{X}$  would be much greater than  $\mathbf{W}$  as SNR grows. Therefore, at high SNR, we might have to train more frequently before  $\Delta_{\mathbf{H}}$  becomes too large.

# Chapter 7

## Summary and Future Directions

### 7.1 Contributions

In this thesis, we studied the problems of efficient designs for multiple antenna communication systems. We studied the design problems in various delay and complexity regimes, from uncoded systems, to structured codes with short delay, to long error correction code enhanced systems.

The main advantages of multiple antennas communication over traditional single antenna communication are the rate gain and robustness gain toward channel fading. To better achieve these gains, we focused on the perspective of diversity-multiplexing tradeoff, a framework established by Zheng and Tse [41], which describes how fast rate increases and how rapidly error probability decays with SNR. We used this diversity-multiplexing tradeoff as a measure of goodness to evaluate systems through out the thesis. As systems become more complex and the code length becomes longer, better tradeoff can be achieved.

In chapter 2, we reviewed the diversity-multiplexing tradeoff framework and also provided some of our own intuitions. We measured the the horizontal spacings between a family of error probability curves and the slopes of these curves to evaluate the diversity-multiplexing tradeoff. This family of curves not only captures the relationship between rate, SNR, and error probability, in the finite SNR regime, but also allows us to see the limiting tradeoff behaviors. We evaluated systems by plot-

ting families of error probability curves and comparing them to the family of outage probability curves.

In section 2.3.5, we briefly discussed the concept of local diversity-multiplexing tradeoff. In certain situations, system designers may care about how the performance of an existing system changes when the operating parameters change slightly, such as when more SNR becomes available, or when the desired data rate increases. In these cases, the local tradeoff is the quantity of interest.

We also discussed the relationship between different segments of the diversity-multiplexing tradeoff curve and the different regions of the  $(\text{SNR}, P_e, R)$  parameter space. Depending on where the system designer wants to operate in, different segments of the tradeoff curve should be focused on. In particular, for higher rates, the segment of the tradeoff corresponds to larger  $r$  is important.

For most of this thesis, we focused on two-transmit two-receive antenna systems, which arises frequently in practice. Even though it is small, this system can provide significant gains over single antenna systems.

In chapter 3, we introduced a lattice-reduction-aided detection idea. By operating traditional low-complexity detectors in a lattice reduced basis, we can achieve near optimal performance with low complexity. In particular, we can achieve the same diversity as the more-complex maximum likelihood detectors, and can achieve the optimal diversity-multiplexing tradeoff achievable by any length-one code. This idea is mainly for low dimensional cases. When extended to higher dimensions, it quickly becomes complex. It can also be used at the transmitter as a pre-coding technique. One main problem with this detector is that it does not treat the boundary of the constellation. Because of this problem, it can not be combined with the tilted-QAM code introduced in chapter 4 to achieve the same tradeoff that ML detectors can.

In chapter 4, we proposed a tilted-QAM code design for the two-transmit two-receive antenna channel that can achieve the optimal diversity-multiplexing tradeoff curve with code length two. This answers the previously open question of whether the optimal tradeoff is achievable at this length. This code improves upon the OSTBC by replacing the repetition used with a rotation, thus avoiding the multiplexing gain



loss. At high SNR, tilted-QAM code is increasing better than OSTBC; however, at low SNR, OSTBC is preferred for its low complexity.

In the tilted-QAM code, one key feature is a set of universally optimal rotation angles that leads to the same worst case determinant for all rates. Similar rotation ideas have been previously studied, but the existence of the universally optimal rotation angles was unknown. This result can also be applied elsewhere, for example, to single antenna communication over multiple fades, as discussed in section 4.6.

We evaluated the performance of the tilted-QAM code from two different perspectives. These error evaluation techniques developed may potentially be used to evaluate other deterministic codes and be extended to higher dimensional cases.

In chapter 5, we investigated using powerful error correction codes in multiple antenna systems to build practical systems with good performance. We explored many schemes and a wide range of possibilities. For low SNR regime, we showed that the low-complexity OSTBC scheme is near optimal. However, for higher SNR levels and systems with more antennas, further research is needed. D-BLAST system is shown to be theoretically optimal but suffers from problems such as error propagation. Other block form variations of BLAST require joint decoder to do well. We also tested system that combine tilted-QAM code with error correction codes. The Hard-decision based system can reach 5 dB from capacity with moderate complexity; while the soft-decision one reduces the gap to 3 dB, but increases the decoding complexity.

In chapter 6, we studied the case of non-coherent communication where neither the transmitter nor the receiver knows the channel. We reviewed several coding designs and a graphical view that relates coding for non-coherent communication to packing on a sphere. By approximating the sphere with tangential planes, we turned the signal design problem into a problem of designing a set of coarse points plus doing coherent communication coding within the planes. We concluded that training scheme corresponds to using just one of the planes and the loss in rate is a constant factor.

## 7.2 Future Directions

### 7.2.1 Coherent Communications

One interesting future direction of research is to extend the tilted-QAM code design to higher dimensions. This is a challenging problem due to the higher number of dimensions and the larger number of variables. One open question we hope to answer is the achievability of optimal diversity multiplexing tradeoff using codes with length  $N_t \leq T < N_t + N_r - 1$ . Gaussian random codes at these lengths are sub-optimal. However, it might be possible to construct deterministic codes that can reach optimality.

Another theoretically interesting topic is the design criteria in higher dimensional cases. In the two-transmit two-receive antenna case, we saw that maximizing the worst-case determinant is important for achieving the optimal tradeoff for  $0 \leq r \leq 1$ , while keeping the degree of freedom and not use repetition is important for achieving the maximum multiplexing gain. In higher dimensional cases, the optimal tradeoff curve has more linear segments, we suspect that there are different criteria for the different segments, and these criteria need to be identified. Also, since the different segments correspond to different  $(\text{SNR}, P_e, R)$  regimes, depending on the operating point, we might need to focus on different criteria.

More practically, if more antennas are available, we can consider using the  $2 \times 2$  tilted-QAM code as a building block. For example, in a  $N_t = N_r = T = 4$  system, if we encode for the four antennas using two independent  $2 \times 2$  tilted-QAM codes and do ML decoding, we expect that a maximum diversity of 8 can be achieved, instead of  $N_t N_r = 16$ . This loss is acceptable practically, since the target error rate needed are usually not too low. More work involving how  $2 \times 2$  tilted-QAM code can be used as a component in systems with more antennas may lead to practical schemes.

Another practical problem that needs to be solved is the joint decoding problem for V-BLAST or X-BLAST. We saw that X-BLAST can achieve optimal performance if joint decoding can be done. Therefore, if joint decoding can be done efficiently, this may lead to many practical applications.

In order to do joint decoding, we must deal with the interference between the transmitted symbols. One low-complexity way to do so is to use the lattice-reduction-aided detectors. We saw earlier that this does not work due to the boundary problem of lattice decoding. Therefore, if we could deal with the boundary efficiently, it might lead to a way of doing joint decoding efficiently and well enough such that the optimal tradeoff can be achieved.

### 7.2.2 Non-coherent Communications

The field of non-coherent communication is still a wide open one. Developing deeper understanding of this problem is very important, because the channel is never known perfectly at the receiver in reality. When systems developed with the channel knowledge assumption are implemented in practice, engineers often have to deal with channel uncertainty issues. If systems could be designed from non-coherent communication theory, they might lead to more robust implementations in practice.

The geometric approach presented in section 6.3 allows existing coherent multiple antenna communication techniques to be applied to the non-coherent case. This may be a promising approach. Directly solving the non-coherent problems seems difficult. Leveraging on existing coherent communication techniques may make the problem easier. One of the key problems that still need to be resolved along this path is how large the tangential planes can be before the approximation of the sphere becomes bad near the edge. This effect also appears in the case of coherent communication with training, in that channel estimation error causes more interference for points at the edge of the constellation. Another key problem is the design of the coarse constellation points. From the size limit of the tangential planes, we can estimate how many coarse constellation points are needed. If this number is small, we may be able to use existing techniques or use very special structures. However, if a large number of points is needed, the design may be difficult.

Among the existing non-coherent communication results, most are developed using the block fading model. This assumption that the channel stays exactly constant could be mis-leading. It allows for techniques such as learning the channel as accu-

rately as desired at the beginning and then treat the channel as coherent, which is not possible if the channel is fading continuously. One consequence of this discrepancy is that the capacity growth rate at high SNR is different depending on the model used.

Another problem associated with the block fading model is the choice of the coherence time  $T$ , the period in which we assume the channel stays exactly constant. If  $T$  were set to be too large, the channel could change significantly within one block, invalidating the block fading assumption. If  $T$  were set to be too small, the channel seen by neighboring blocks would be quite close, and not taking advantage of this correlation is inefficient.

One non-coherent signal design that does not use the block fading model and avoids the above problem is the differential coding scheme. It uses small blocks and assumes the channel does not change much between neighboring blocks. Slight channel variations are treated as noise. This scheme may be further improved by considering the channel variation across blocks.

Generally, further research based on more accurate continuous fading models is needed to understand its effect and how good block fading model really is.

In terms of theoretical results, one that is useful but may be difficult to develop is the capacity achievable at low SNR. This result is important for evaluating system performance in practical regimes.

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